Mesh Parametrization via Simulation of Central Force Field Motion

Gábor Fábián
ELTE-IK, Department of Numerical Analysis
robagnaibaf@gmail.com

Introduction

Parametric surfaces play an important role in computer-aided geometric design. Parametric surfaces are defined by continuous functions (parametrizations), contrary the surfaces in computer graphics are represented with discrete vertices and surface elements. Any parametrization can be easily transformed into a discrete model, in opposite direction the problem is more difficult. In this research we are looking for a parametrization of triangular meshes that are topologically equivalent to the sphere. This problem is called mesh parametrization, it can be used in many areas of computer graphics, e.g. detail mapping, morphing, mesh editing, recovering, remeshing, compression, surface fitting, etc. For further details see [6].

Interpretation of Surfaces

In **differential geometry** a set $S \subset \mathbb{R}^3$ is a parametric surface, if there exists a bijection $\phi : I \times J \to S$ between some 2-dimensional interval and S [1].
In **topology** a surface is a 2-manifold, i.e. a second countable Hausdorff-space such that every point in S has a neighborhood which is homeomorphic to some open subset in $\mathbb{R}^2$ [3].
In **computer graphics** surfaces are usually identified with its boundary representation. For example a triangular mesh is defined by its vertices and faces. Let $\mathcal{M} := (V, F)$ be a triangular mesh with

$$V : \{1, \dots, m\} \to \mathbb{R}^3$$
$$F : \{1, \dots, n\} \to \{1, \dots, m\}^3 \quad (m, n \in \mathbb{N})$$

The j-th vertex of the i-th face is denoted by v_{ij}, the i-th surface element is defined by the convex combination of the v_{ij} vertices:

$$\triangle_i^{\mathcal{M}} := \left\{ \sum_{j=1}^{3} \alpha_j v_{ij} \;\middle|\; \sum_{j=1}^{3} \alpha_j = 1, \quad \alpha_j \geq 0 \right\}$$

Then the surface defined by the mesh $\mathcal{M}$ is

$$\partial\mathcal{M} := \bigcup_{i=1}^{n} \triangle_i^{\mathcal{M}}$$

References

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[5] A. A. G. Requicha *Mathematical Models of Rigid Solid Objects*, Production Automatation Project (1977).

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Parametrization of Meshes

The topological properties of meshes as connectedness, compactness and orientability can be computed easily [5]. **Classification theorem of surfaces** states that *every connected, compact, orientable 2-manifold is equivalent to the sphere or the connected sum of tori.* As mentioned before, a mesh $\mathcal{M}$ is a polygonal presentation of a surface $\partial\mathcal{M}$ with finitely many vertices, edges and faces. This allows us to test that a mesh is homeomorphic to the unit sphere or not. The **Euler characteristic** of a mesh can be computed by

$$\chi = m - l + n$$

where $m, l, n \in \mathbb{N}$ are the numbers of vertices, edges and faces, respectively. The Euler characteristic of a compact surface, which is topologically equivalent to the sphere equals to 2. Now, let us suppose, that the mesh surface $\mathcal{M}$ is connected, compact, orientable 2-manifold, which is topologically equivalent to the sphere. Then a parametrization of $\partial\mathcal{M}$ can be computed as follows. The surface is topologically equivalent to the sphere, so there exists a homeomorphism $H : \partial\mathcal{M} \to \mathbb{S}^2$. The following function $G : [0, 2\pi) \times [0, \pi) \to \mathbb{S}^2$ is a parametrization of the unit sphere

$$G(\theta, \phi) := (\sin\theta\cos\phi, \;\; \sin\theta\sin\phi, \;\; \cos\theta)$$

Since $H^{-1} : \mathbb{S}^2 \to \partial\mathcal{M}$ is a bijection

$$H^{-1} \circ G : [0, 2\pi) \times [0, \pi) \to \partial\mathcal{M}$$

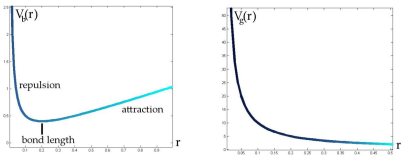
is a parametrization of the surface $\partial\mathcal{M}$. Although, the existence of the homeomorphism H can be easily proved, it is usually hard to determine exactly. We used a simulation of a particle system defined by the model vertices to determine the desired homeomorphism.

Physical Simulation

In our research we simulated physically to „blow up" the mesh like an air balloon. Let us define the following central potentials:

$$V_b(|r|) := ed^2\frac{1}{|r|} + e|r| \quad V_g(|r|) := \frac{g}{\alpha - 1}\frac{1}{|r|^{\alpha-1}}$$

where e, d, g, α are constants $r = |r|\hat{r} \in \mathbb{R}^3$ vector. V_b simulates the bonds between the adjacent vertices of the surface, V_g simulates a generalized gravitational potential.



The force is defined as the negative gradient of the potential [2].

$$F_b(r) := \left(ed^2\frac{1}{|r|^2} - e\right)\hat{r} \quad F_g(r) := -g\frac{1}{|r|^{\alpha}}\hat{r}$$

Let c be the center of the gravitational force, x_i is the position of the i-th vertex. Then the force that acting on the i-th vertex can be computed as

$$F_i = \sum_{<ij>} F_b(x_i - x_j) + F_g(x_i - c)$$

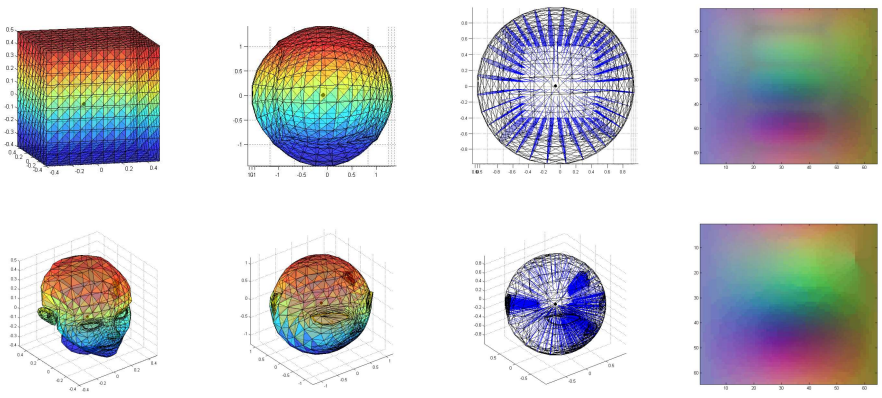
where the sum is over the adjacent vertices. The equations of motion can be obtained using Newton's Second Law:

$$F_i = m_i\ddot{x}_i$$

where m_i is the mass of the i-th vertex. We simulated the motion using Euler method [4].

Experiments

In our experiments F_b between vertices i, j was defined using $d = |V(i) - V(j)|$, therefore without F_g the system is at rest. Thereafter the system with a negative (repulsive) gravity was perturbed. The experiments were supervised, the parameters were adapted to the current state of the mesh.



The initial system is the original mesh *(1st column)*. After a successful simulation *(2nd column)* the mesh is sufficiently similar to a sphere, therefore we can project it onto the unit sphere avoiding overlapping spherical triangles. Finally H can be defined by the difference between the initial and the final position of the vertices *(3rd column)*, in this way we can determine the $H^{-1} \circ G$ parametrization of the mesh *(4th column)*, where $\mathbb{R}^3$ vectors are converted to colors.