Minimum Arc Length Curves & Optimization in Spline Interpolation

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Introduction

Let us consider a spacecraft in the two dimensional plane, which can be moved in any direction from a given start point. Let us suppose that some sequence of points are given, and the points should to be visited in a given order, after that the spacecraft should to return to the start. We assume, that the position and the velocity functions of time are continuous. How can we move the spacecraft to reach the minimum distance travelled? The same problem in mathematical formulation sounds like: how can we define a differentiable closed planar curve, that interpolates in given points and it has minimum arc length? In this research we give a particular solution to this problem using periodic spline interpolation and optimization.

Splines & Boundary Conditions

Let $a, b \in \mathbb{R}$ where $a < b$ and $\Omega_n := \{x_0, \dots, x_n\}$ such that $a = x_0$, $b = x_n$ $x_i < x_{i+1}$. Consider an $f \in C[a,b]$ function, which has a period of $x_n - x_0$. Then obviously $f(x_0) = f(x_n)$. The cubic spline interpolant of set of points $(x_0, f(x_0)), \dots, (x_n, f(x_n))$ is an $s \in C^2(a,b)$ function, such that [4]

$$s_{|[x_i,x_{i+1}]}(x) := p_i(x) \qquad (i = 0, \dots, n-1)$$

$$p_i \in \mathcal{P}_3 := \left\{ \sum_{k=0}^{3} a_k x^k \mid a_k \in \mathbb{R} \right\}$$

$$\begin{aligned} s(x_i) &= f(x_i) && (i = 0, \dots, n) \\ p_i'(x_{i+1}) &= p_{i+1}'(x_{i+1}) && (i = 0, \dots, n-2) \\ p_i''(x_{i+1}) &= p_{i+1}''(x_{i+1}) && (i = 0, \dots, n-2) \end{aligned}$$

Then we say, s is cubic spline on Ω_n, in notation $s \in S_3(\Omega_n)$. It is known, that the cubic spline interpolation problem is well-posed with the common periodic boundary conditions: $s'(x_0) = s'(x_n)$ and $s''(x_0) = s''(x_n)$ [4]. In our case, the boundary conditions have to be weakened, because there is no parameter that can be used for optimization. We would not require the continuity of the second derivative (in terms of periodicity), so we will use the following boundary conditions instead: $s'(x_0) = s'(x_n)$ and $s''(x_0) = \beta$. Now the β is a free parameter of the model, therefore we are able to look for optimal solutions of various problems. Without loss of generality, we can consider $\mathbb{T}$ the one dimensional torus instead of $[a,b]$ interval. The functions from the space $C^2(\mathbb{T} \setminus \{0\}) \cap C^1(\mathbb{T})$ are in the focus of our interest, that are continuously differentiable on the torus and twice continuously differentiable on $\mathbb{T}$, except the origin. Consider a functional: $I : C^2(\mathbb{T} \setminus \{0\}) \cap C^1(\mathbb{T}) \to \mathbb{R}$. Now we can formulate our problem. Let us suppose, that $s_\beta \in S_3(\Omega_n)$ and it interpolates the $f \in C(\mathbb{T})$ function at the points of Ω_n, moreover $s_\beta'(x_0) = s_\beta'(x_n)$ and $s_\beta''(x_0) = \beta$. Then we are looking for β^* the minimizer of functional, i.e. $\beta^* = \operatorname{argmin}_{\beta \in \mathbb{R}} |I(s_\beta)|$ [2]. We are able to determine the entire spline function given with the modified boundary conditions.

Optimization Problem in Multiple Dimensions

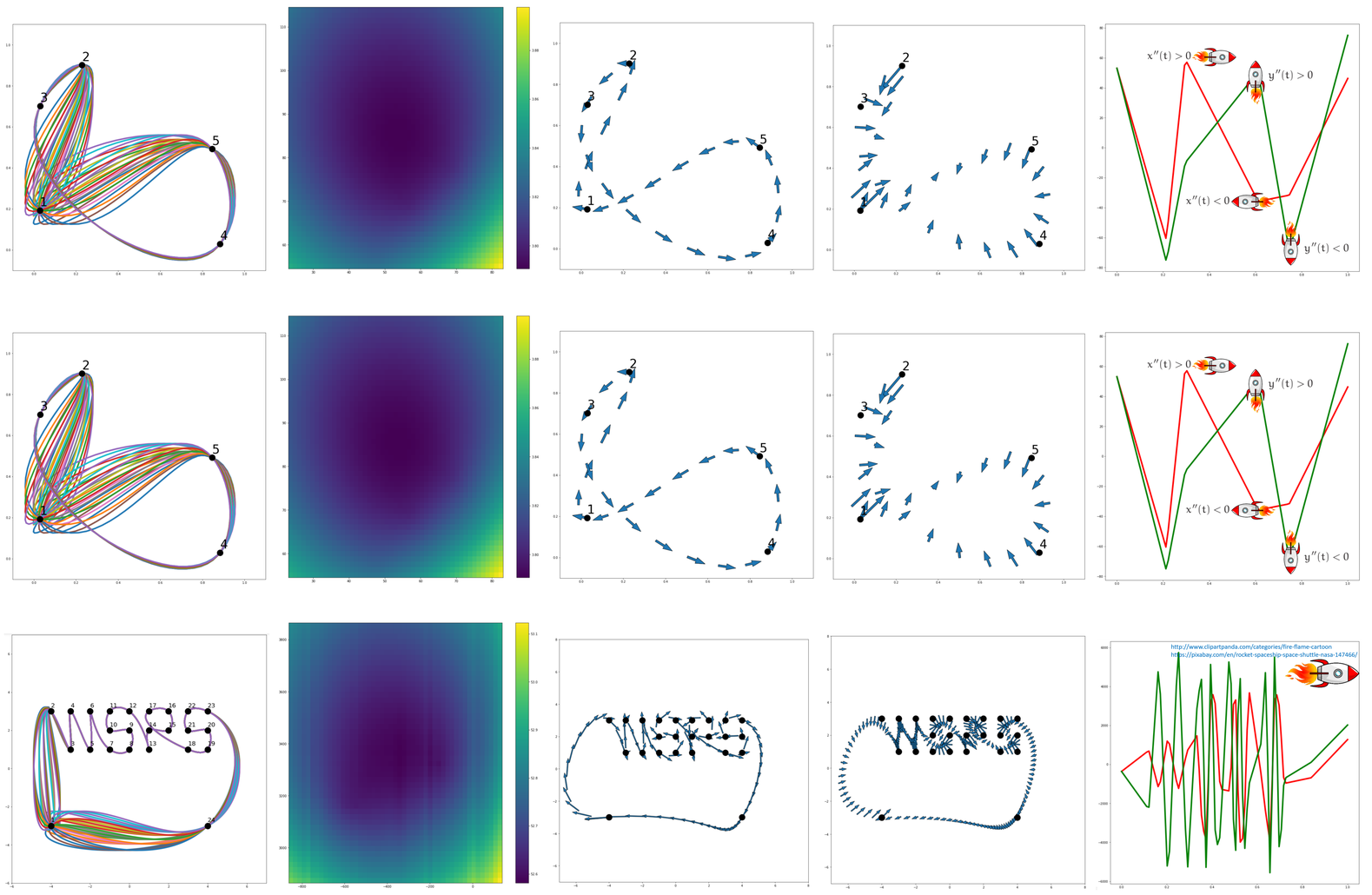
We can easily generalize our problem to multiple dimensions. Assume that $x_0, x_1, \dots, x_n \in \mathbb{R}^d$ are arbitrary points in $\mathbb{R}^d$. Then we can construct a differentiable curve $\Gamma : \mathbb{T} \to \mathbb{R}^d$ which interpolates at the given points using the periodic cubic spline interpolants of the coordinate functions of the curve: $\Gamma(t) = (\Gamma_1(t), \dots, \Gamma_d(t))$, $\Gamma_j(t) := s_{j,\beta_j}(t)$ where $s_{j,\beta_j}(t_i) = (x_i)_j$ and $t_i = \frac{\sum_{j=0}^{i-1} \|x_{i+1} - x_i\|}{\sum_{j=0}^{n-1} \|x_{i+1} - x_i\|}$. Modifying the functional we obtain the optimization problem in multiple dimensions:

$$I : (C^2(\mathbb{T} \setminus \{0\}) \cap C^1(\mathbb{T}))^d \to \mathbb{R} \qquad \mathbb{R}^d \ni \beta^* = \operatorname{argmin}_{(\beta_1, \dots, \beta_d) \in \mathbb{R}^d} |I(s_{1,\beta_1}, \dots, s_{d,\beta_d})|$$

Now let us suppose, we have a two dimensional polygon with some vertices (x_i, y_i), $(i = 0, \dots, n)$ where $(x_0, y_0) = (x_n, y_n)$. Now we are looking for the Γ differentiable planar curve *with minimum arc length*, that interpolates at the given points. We can determine the coordinate functions of Γ using the periodic cubic spline interpolants of (t_i, x_i) and (t_i, y_i) point sets: $\Gamma(t) = (s_{1,\beta_1}(t), s_{2,\beta_2}(t))$ where $s_{j,\beta_j}(t_i) = (x_i)_j$. The arc length of Γ plane curve can be calculated as $\int_0^1 L(t)dt$ [1] where $L(t) = \sqrt{s_{1,\beta_1}^2(t) + s_{2,\beta_2}^2(t)}$. It is almost impossible to calculate this integral in general, but we can use the trapezoidal rule [3] to approximate the solution. Let us fix a great natural number $M \in \mathbb{N}$. Then $\int_0^1 L(t)\,dt \approx \sum_{i=0}^{M-1} \frac{1}{2M} \left(L(\frac{i+1}{M}) + L(\frac{i}{M}) \right)$. This way the two-dimensional optimization problem can be written in the following form: $\beta^* = \operatorname{argmin}_{(\beta_1,\beta_2) \in \mathbb{R}^2} \frac{1}{2M} \sum_{i=0}^{M-1} \left(L(\frac{i+1}{M}) + L(\frac{i}{M}) \right)$. The splines for fixed (β_1, β_2) can be determined, the arc length can be approximated, after that arbitrary algorithm can be used to find the (β_1^*, β_2^*) local minimizer of the arc length.

Results

Let us denote the minimum arc length parameterizations with $x(t) := s_{1,\beta_1^*}(t)$ and $y(t) := s_{2,\beta_2^*}(t)$. In the terminology of the original problem, on the optimum orbit the position of the spacecraft at the time t is $(x(t), y(t))$, therefore its velocity and acceleration can be expressed as $(x'(t), y'(t))$ and $(x''(t), y''(t))$, respectively. Since the splines are known, we can easily obtain the acceleration which is function of time. By Newton's Second Law [5] we know, that $(F_x(t), F_y(t))/m = (x''(t), y''(t))$, consequently we can calculate the forces at a given moment, that should be acting on the spacecraft to reach the desired motion, let us say, the control function. In the figures we can see from left to right: (1) some of the possible trajectories of a given point set, (2) the arc length as funtions of (β_1, β_2), (3) the velocity and (4) acceleration function of the spacecraft on the points of the minimum arc length trajectory, finally (5) the coordinate functions of the control function.



[1] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov. *Modern Geometry - Methods and Applications*. Springer, 1992.

[2] J. Nocedal, S. J. Wright. *Numerical Optimization (Second Edition)*. Springer, 2006.

[3] W. H. Press et al. *Numerical recipes in C - The Art of Scientific Computing*. Cambridge University Press, 1992.

[4] J. Stoer, R. Bulirsch. *Introduction to Numerical Analysis (Second Edition)*. Springer-Verlag, 1993.

[5] L. D. Landau, E. M. Lifshitz *Course of Theoretical Physics Volume 1. Mechanics - Third Edition*, Elsevier (1976).