FAST AND LIGHTWEIGHT METHOD FOR MORPHING GENUS-0 MESHES

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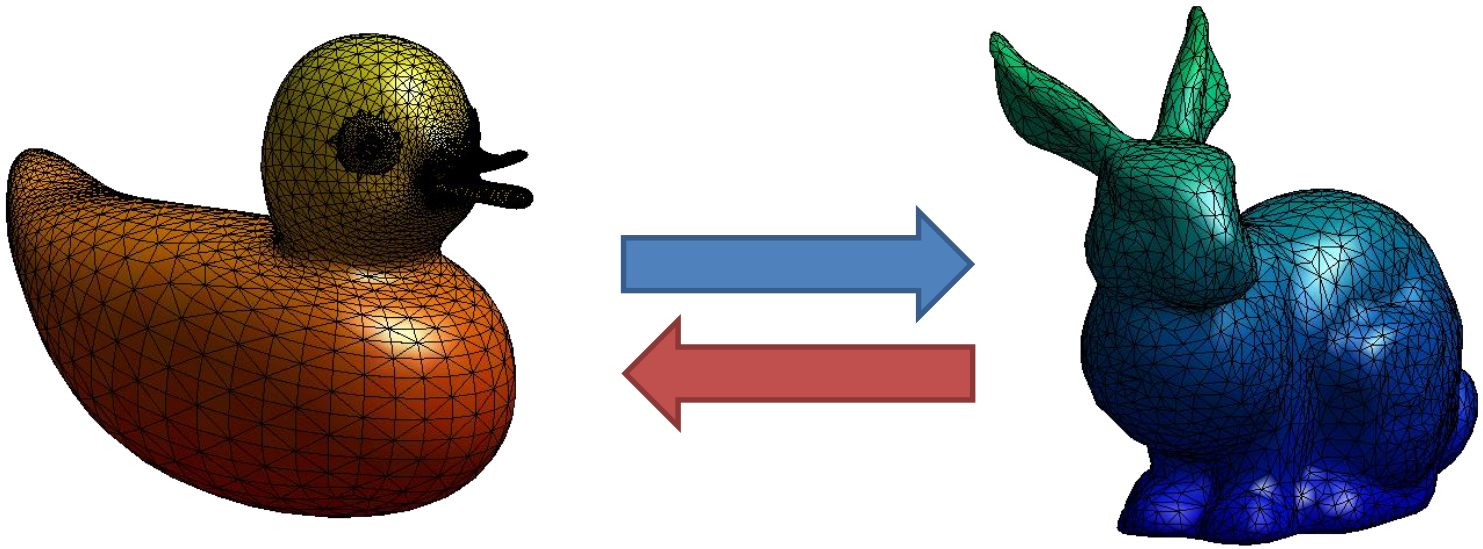
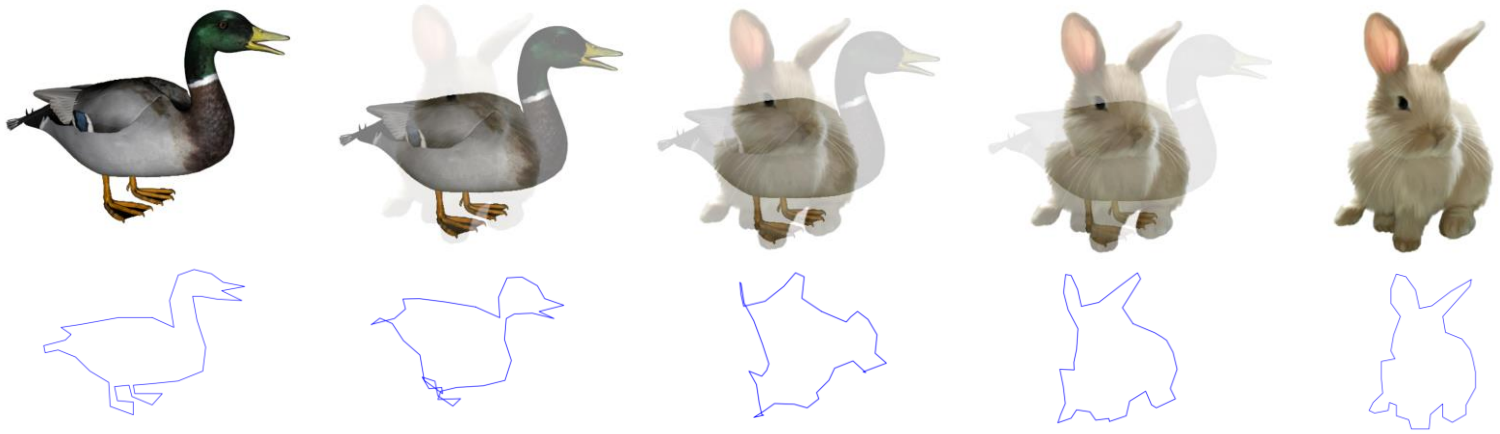
PROBLEM

Let us suppose, that two 3D objects are given.

- How can we define a ,,smooth transition'' between them?
- What does „smooth transition" means?
- Does it exist at all?
- What is the similar in the two objects?
  - Ducky: 6390 vertices, 12776 triangles
  - Bunny: 2503 vertices, 5002 triangles

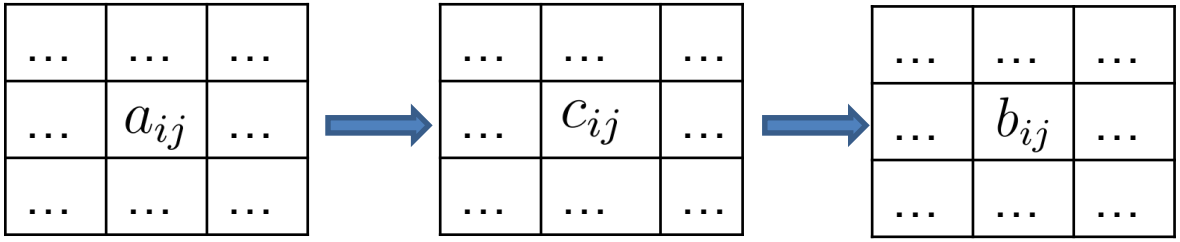


A SIMILAR PROBLEM FROM 2D



Why is it easier in case of images and polygons?

$$A, B, C \in \mathbb{R}^{n \times n}$$



$$c_{ij} = (1-t)a_{ij} + tb_{ij} \qquad (t \in [0,1])$$

Source of the images:
http://www.pngmart.com/image/2572
http://www.pngmart.com/image/5044

TRIANGLE MESH REPRESENTATION

Triangle mesh can be imagined as a collection of triangle elements, a mesh can be defined by the set of vertices and the indices of the triangles.

$$(V, T)$$

$$V \in \mathbb{R}^{m \times 3} \qquad (v_{i1}, v_{i2}, v_{i3}) \in \mathbb{R}^3$$

$$T \in \mathbb{N}^{n \times 3} \qquad (t_{k1}, t_{k2}, t_{k3}) \in \mathbb{N}^3$$

The edges can be generated from the triangle indices, therefore a triangle mesh can be imagined as a graph, as well.

$$E \in \mathbb{N}^{l \times 2}$$

$$t_k = (p, q, r) \rightarrow \begin{cases} (p, q) = (e_{\alpha 1}, e_{\alpha 2}) \\ (q, r) = (e_{\beta 1}, e_{\beta 2}) \\ (r, p) = (e_{\gamma 1}, e_{\gamma 2}) \end{cases}$$

ADJACENCY MATRIX AND DEGREE MATRIX

Simplifying the formalism: discrete interval and circular indexing of triples

$$[a..b] := [a, b] \cap \mathbb{N}$$

$$t_{p,q} := t_{p,((q-1) \bmod 3)+1}$$

(Directed) adjacency matrix:

$$A \in \mathbb{R}^{m \times m}$$

$$a_{ij} := |\{p \in [1..m] \mid \exists q \in [1..3] \quad : \quad t_{pq} = i \ \wedge \ t_{p,q+1} = j\}|$$

(Out) degree matrix:

$$D \in \mathbb{R}^{m \times m}$$

$$d_{ij} := \begin{cases} \sum_{k=1}^{m} a_{ik} & i = j \\ 0 & \text{otherwise} \end{cases}$$

TOPOLOGICAL PROPERTIES OF TRIANGLE MESHES I.

- If a triangle mesh is triangulation of a topological **2-manifold**, then

$$\forall i, j \in [1..m] \quad : \quad a_{ij} + a_{ji} \leq 2$$

- If a triangle mesh is triangulation of a **closed** topological **2-manifold**, then

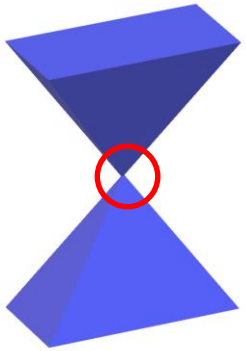
$$\forall i, j \in [1..m] \quad : \quad a_{ij} + a_{ji} \in \{0, 2\}$$

- If a triangle mesh is triangulation of an **orientable closed** topological **2-manifold**, then

$$A^T = A \quad \wedge \quad \forall i, j \in [1..m] \quad : \quad a_{ij} \in \{0, 1\}$$

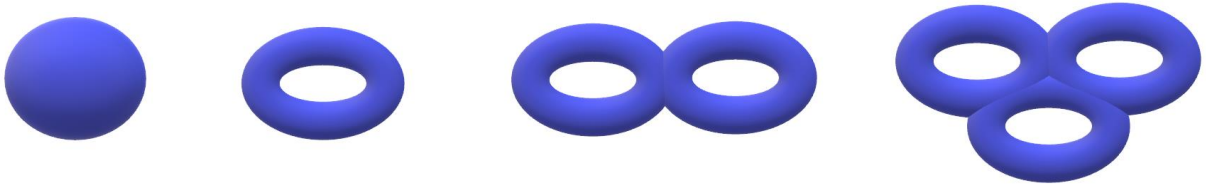
- If a triangle mesh is triangulation of an **orientable closed** topological **2-manifold**, then it is **connected** iff *A* is irreducible.

- + if for every vertex of a triangle mesh adjacent to a unique ,,triangle fan'', then the mesh is triangulation of a orientable closed topological 2-manifold.
  (This statement can be formalized more rigorously, but now it is enough, that quite easy to verify this property algorithmically)



TOPOLOGICAL PROPERTIES OF TRIANGLE MESHES II.

- **Classification theorem of 2-manifolds:** If a triangle mesh that is a triangulation of a closed connected orientable 2-manifold, then it is topologically equivalent to the sphere, or the n-tori.



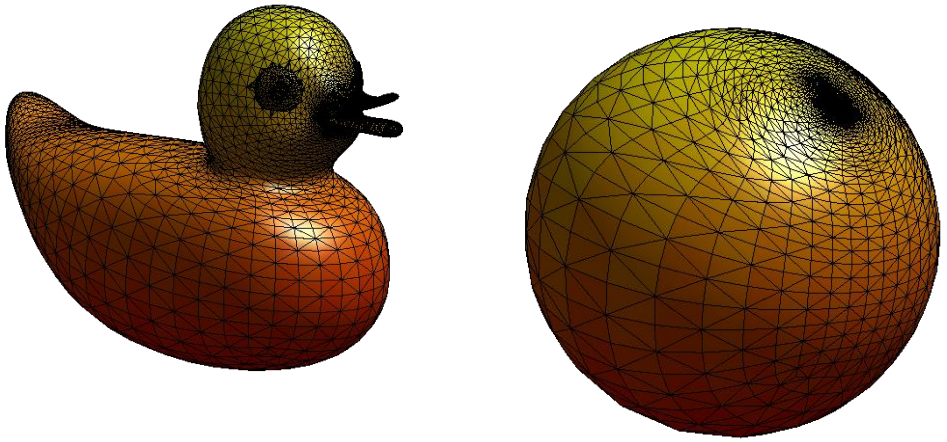
- Topological equivalence means, that exists a continuous and inverse continuous bijection between the two manifolds, i.e. a continuous deformation between them (continuity=„no cutting”, inverse continuity=„no gluing”).

- **Euler-characteristic:** $\chi = m - l + n$

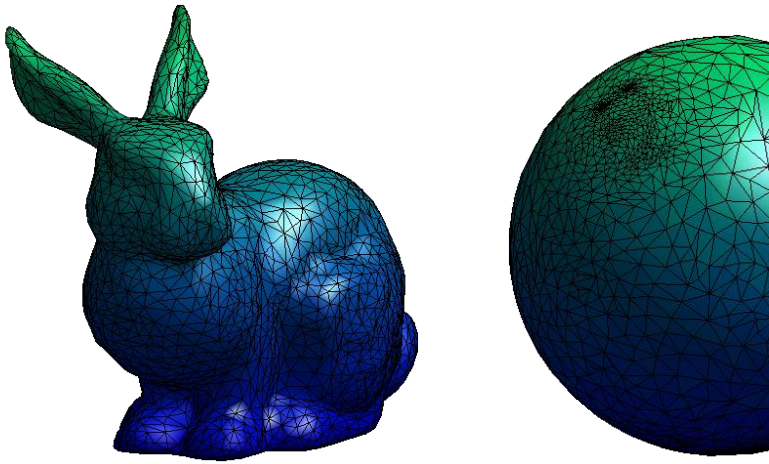
- A closed connected orientable 2-manifold is topologically equivalent to the sphere, iff its Euler-characteristic equals to 2. Furthermore, the Euler characteristic of a sphere is 0, for the torus it is 1, and generally the g-torus has an Euler-characteristic of 2-2g, where the g number called genus. That’s why we call genus-zero meshes, the meshes that are topologically equivalent to the sphere.

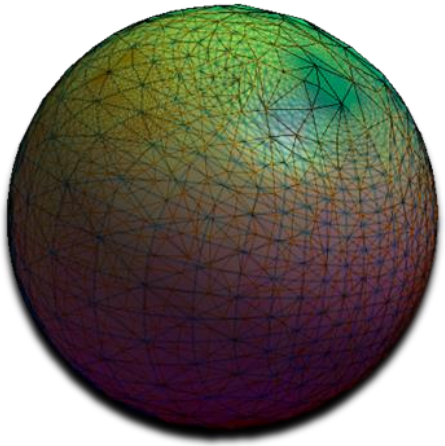
ABOUT SIMILARITY

$$\chi = 6390 - 19164 + 12776 = 2$$



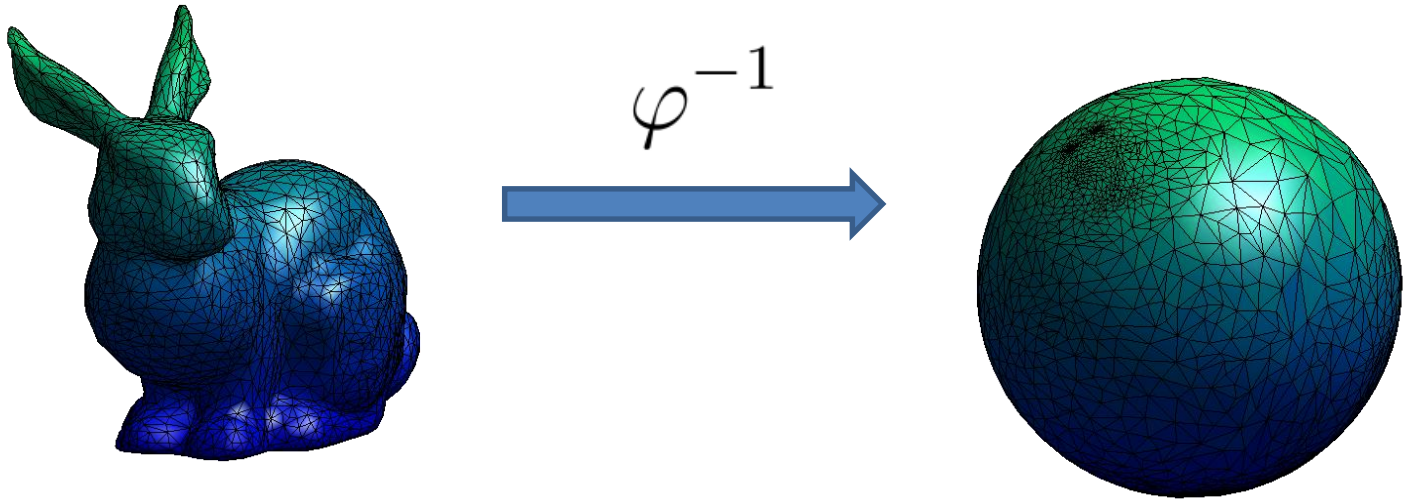
$$\chi = 2503 - 7503 + 5002 = 2$$





SPHERICAL PARAMETERIZATION

- It is easy to prove, that a homeomorphism exists between them, but it is quite hard to determine it.



- How can we do it? A possible answer is: via a conformalized mean curvature flow (For details, see [2] and [7]).

CONFORMALIZED MEAN CURVATURE FLOW

- What is a Conformalized Mean Curvature Flow? (CMCF)
  - It is an iteration method…

$$V^{(k+1)} = (M - \delta L)^{-1} M V^{(k)}$$

  - …with a trick to avoid singularity

$$d^{(k)} := \frac{\sum_{j=1}^{n} \Delta_j^{(k)} c_j^{(k)}}{\sum_{j=1}^{n} \Delta_j^{(k)}} \qquad v_i^{(k)} := \frac{v_i^{(k)} - d^{(k)}}{\sqrt{\sum_{j=1}^{n} \Delta_j^{(k)}}}$$

$v_i^{(k)}$ is the momentary position of the i-th vertex

$c_i^{(k)}$ is the barycenter of the i-th triangle

$\Delta_i^{(k)}$ is the area of the i-th triangle

$L = A - D$ the Laplacian of the mesh

M the mass matrix of the mesh: m_{ii} is the third of the area of the triangles adjacent to the i-th vertex

GENERIC MESH MORPHING SCHEME

- Most mesh morphing methods works on genus-zero closed connected orientable 2-manifolds. *(Complex meshes are often partitioned to genus-zero meshes with boundary)*

The main steps of a mesh morphing algorithm are:

0. Normalize the pose of the meshes. (Feature matching)
1. Compute spherical parameterizations of source and target meshes.
2. Create an overlapping map. (Feature alignment)
3. Construct a supermesh. (Mesh simplification)
4. Interpolate between source and target vertex positions.

The problem is creating an overlapping map, and construct a supermesh from a spherical parameterization is often hard.

On the other hand, constructing a supermesh from a planar parameterization is much more easier.

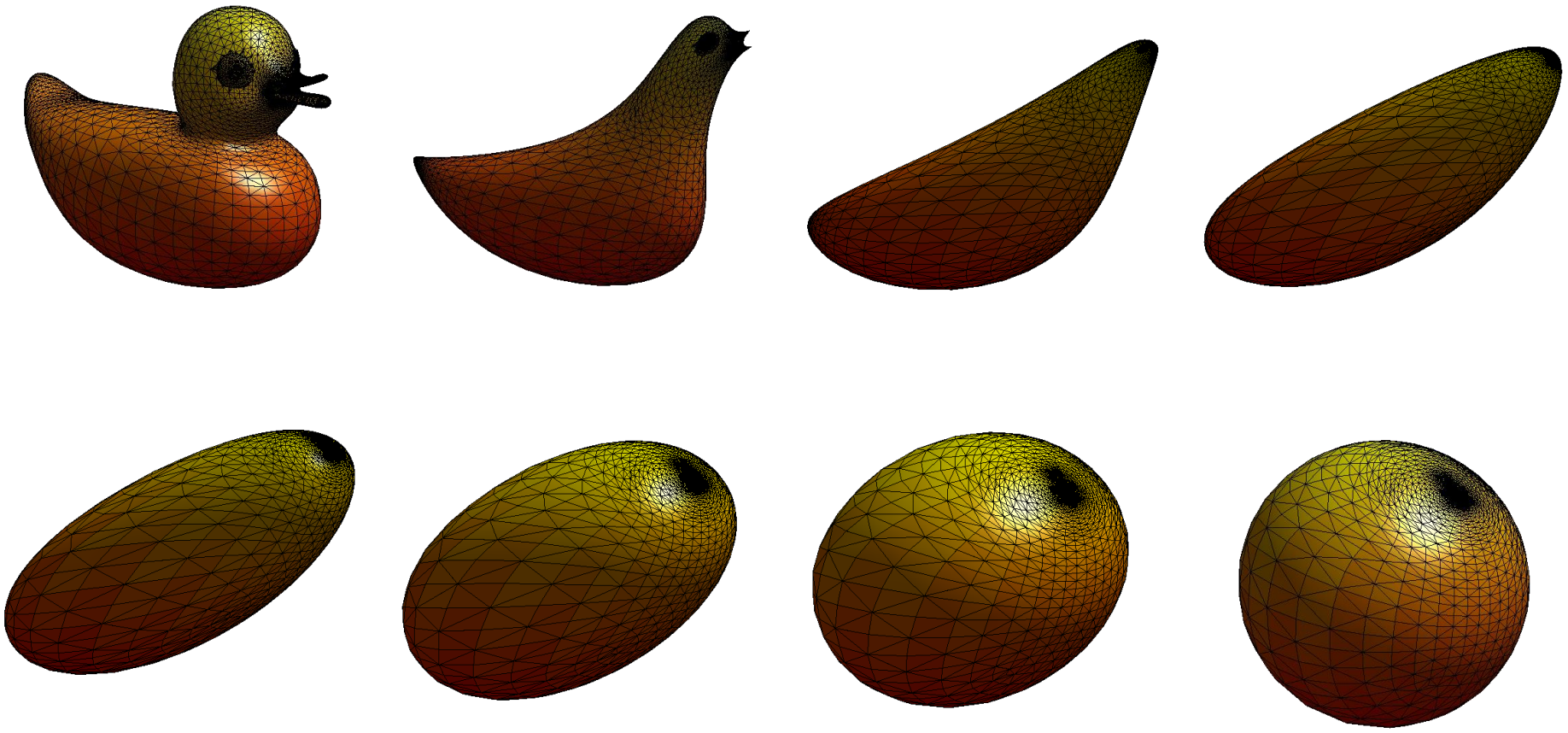
SIMPLIFIED MESH MORPHING SCHEME

- Most mesh morphing methods works on genus-zero closed connected orientable 2-manifolds. *(Complex meshes are often partitioned to genus-zero meshes with boundary)*

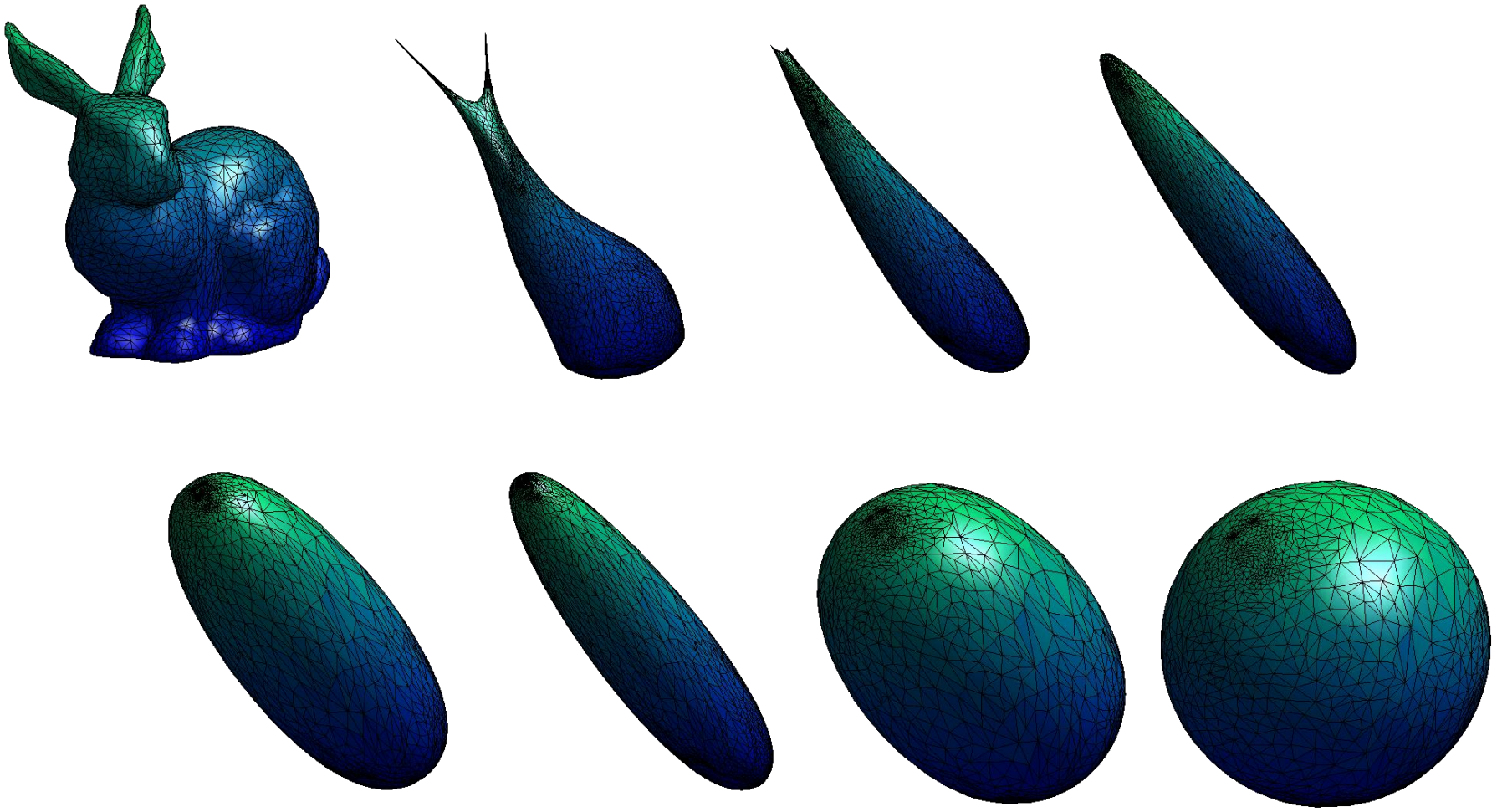
The main steps of a mesh morphing algorithm are:

0. Normalize the pose of the meshes. (Feature matching)
1. Compute spherical parameterizations of source and target meshes.
2. Compute stereographic projections of the points of the spherical parameterizations. (Feature alignment)
3. Delete all edges from the projected (planar) parameterezitations, and compute the (Delaunay-) triangulation of the joint point set.
4. Compute the new vertex positions of the source and target meshes (on the supermesh) by a scattered interpolation technique. (Mesh simplification)
5. Interpolate between source and target vertex positions.

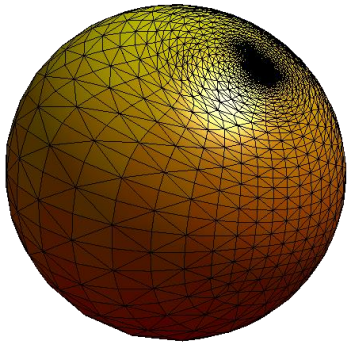
CONFORMALIZED MEAN CURVATURE FLOW I.

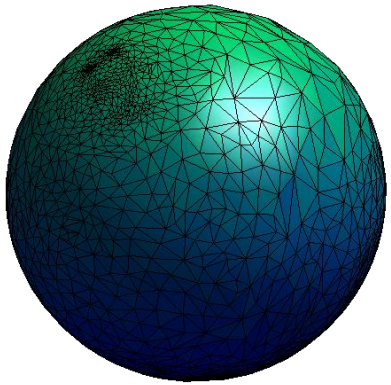


CONFORMALIZED MEAN CURVATURE FLOW II.

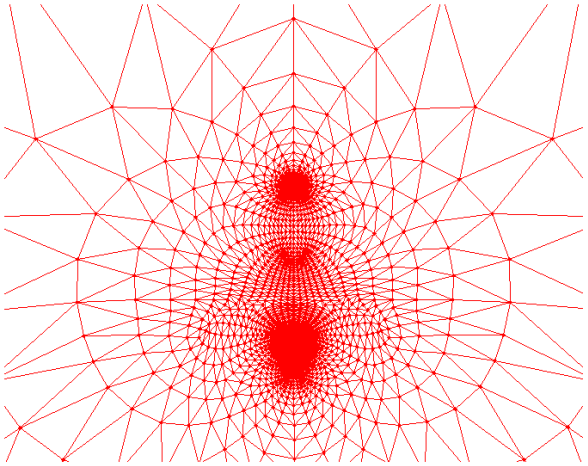


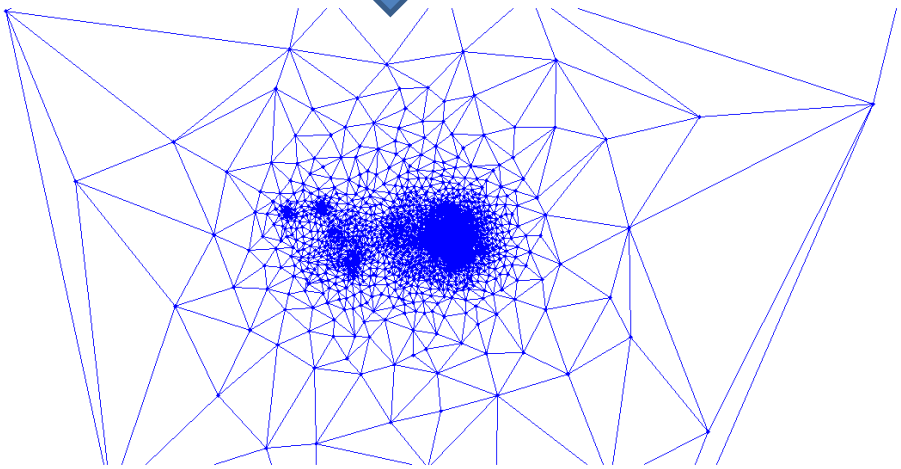
STEREOGRAPHIC PROJECTION OF SPHERICAL PARAMETERIZATION'S POINTS



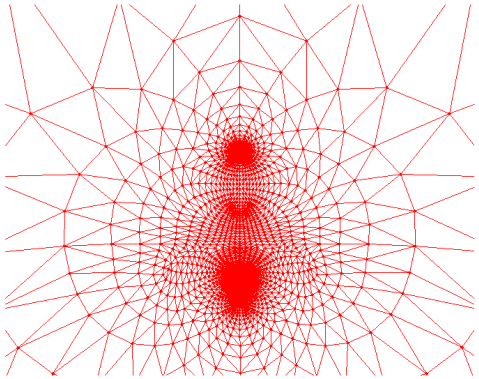


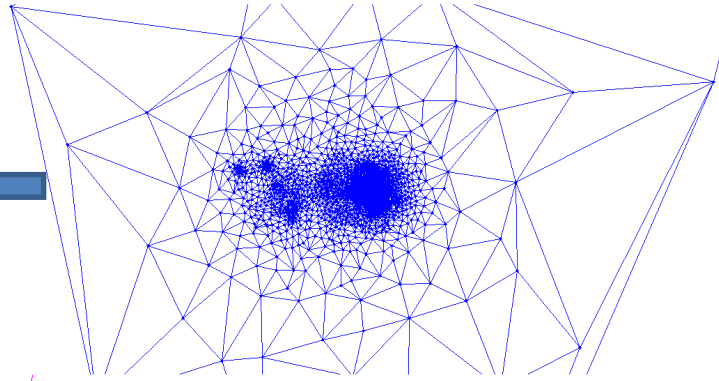
$$\pi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

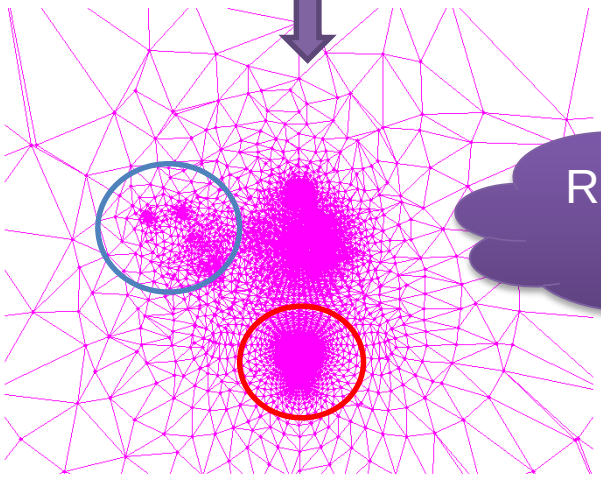




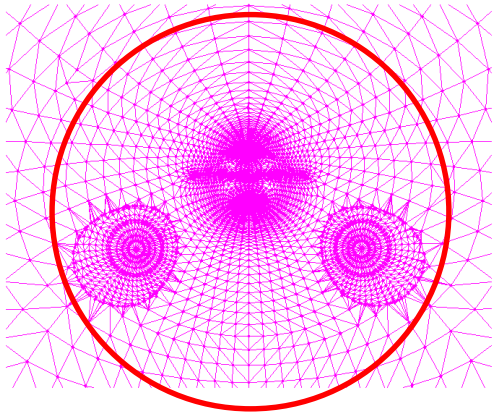
CONSTRUCTING (PLANAR) SUPERMESH

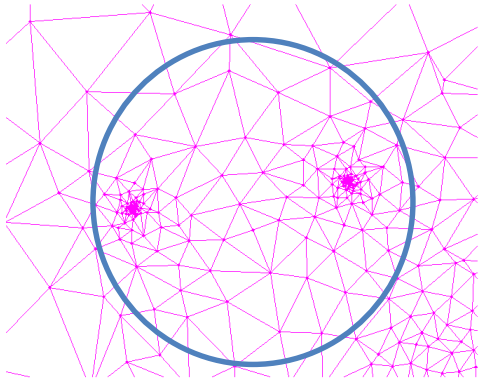




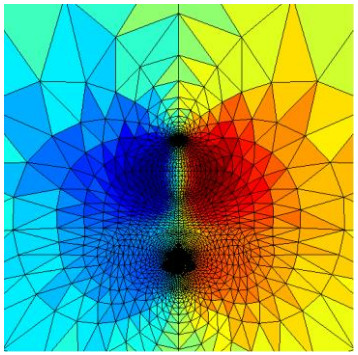
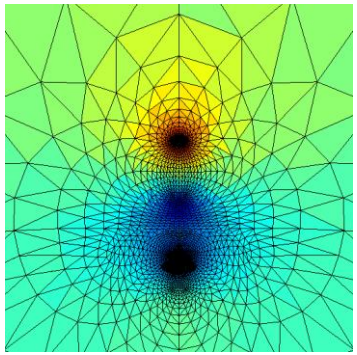
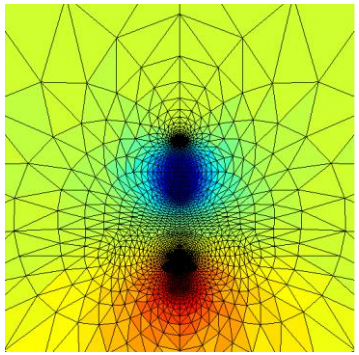
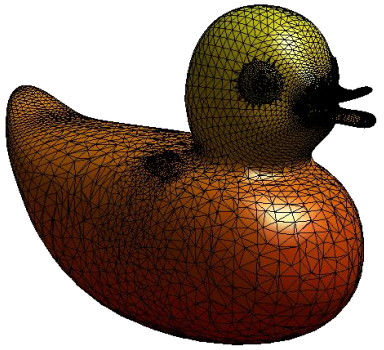
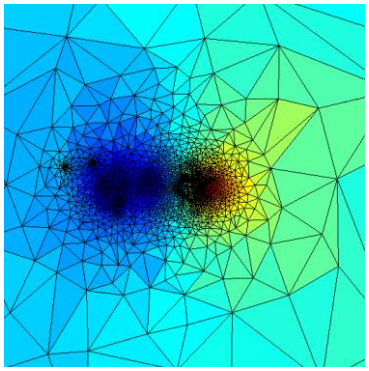
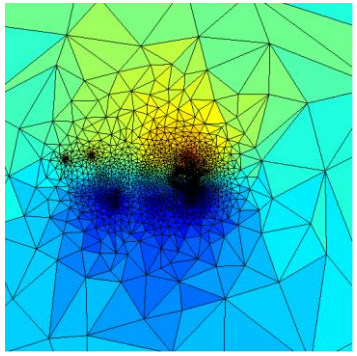
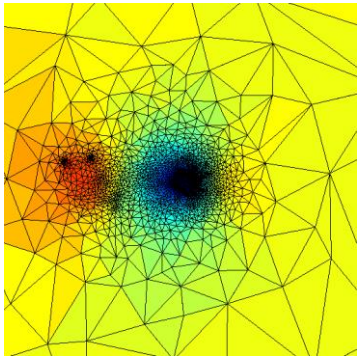
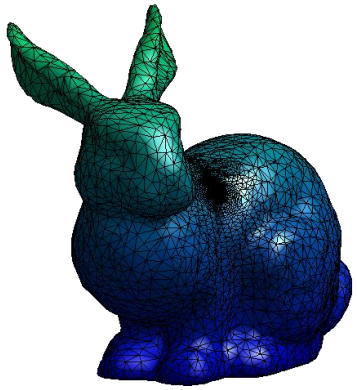


Remove all edges
+ retriangulate

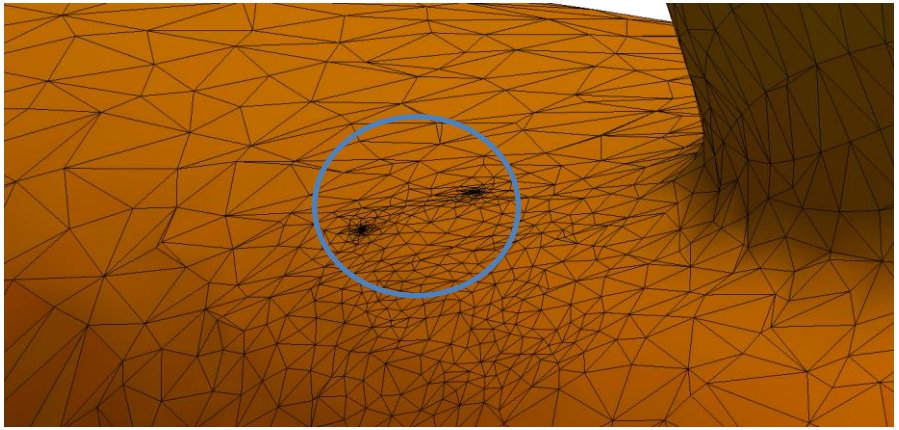


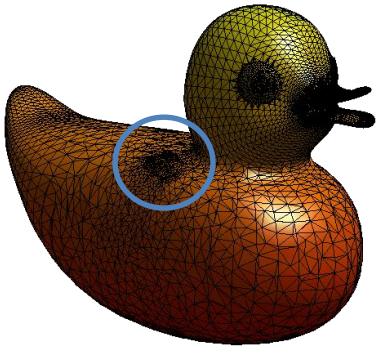


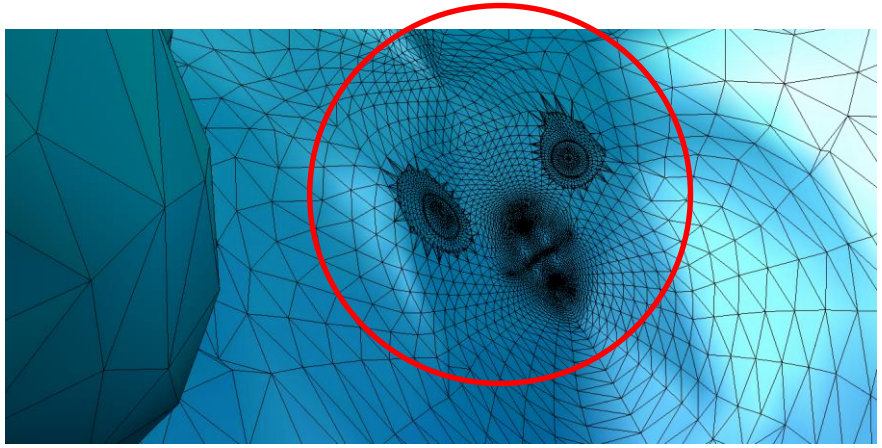
COMPUTE OVERLAID PARAMETERIZATIONS VIA SCATTERED DATA INTERPOLATION

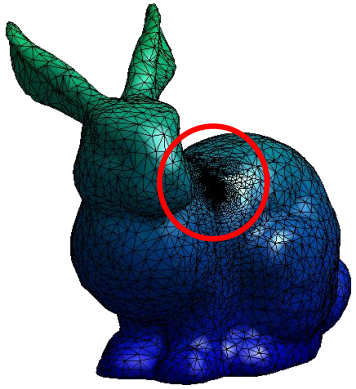









RECONSTRUCTION









MESH MORPHING

$$(1-t)V_S + tV_T \quad (t \in [0,1])$$

OPEN QUESTIONS, FURTHER PLANS

- The convergence of the CMCF for genus-zero meshes have not been proved yet, the hypothetical convergence rely only on empirical facts.

- Why don’t we leave CMCF, and why don’t we work immediately with planar parameterizations, as Tutte-embedding? Planar parameterizations generated by barycentric embeddings are very unbalanced, the inhomogeneous distribution of the vertices is a big problem by mesh morphing.

- Simultaneous usage of scattered data interpolation and retriangulation of the parameterization can cause unwanted effects by reconstruction, it is easy to give examples, when the reconstructed mesh will be self-intersecting.

- It is an interesting question, that how should be possible (by moving, adding, removing vertices) to achieve the Delaunay-property of the point set of planar parameterization. In other words: is it possible to construct a planar parameterization, on which the Delaunay retriangulation process is identity?

- Various applications: mesh simplification, mesh repairing, mesh compression, etc.

REFERENCES

[1] M. S. Floater, K. Hormann. Surface Parameterization: a Tutorial and Survey. *Advances in Multiresolution for Geometric Modelling*, 157--186, 2005.

[2] M. Kazhdan, J. Solomon, M. Ben-Chen. Can mean-curvature ow be made non-singular? *Computer Graphics Forum*, 31 (5):1745--1754, 2012.

[3] J. M. Lee. *Introduction to Topological Manifolds*. Springer, 2011.

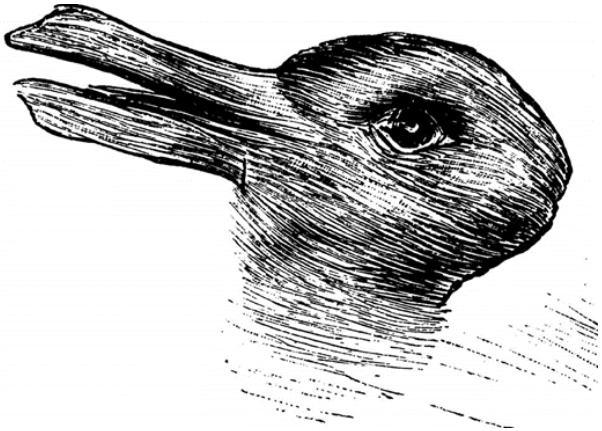
[4] B. C. Mocanu. 3D mesh morphing. *Economies and finances. Institut National des Télé-communications*, 2012.

[5] C. Peng, S. Timalsena. Fast mapping and morphing for genus-zero meshes with cross spherical parameterization. *Computers & Graphics*, 59: 107--118, 2016.

[6] E. Praun and H. Hoppe. Spherical parametrization and remeshing. *ACM Transactions on Graphics* 22 (3): 340--349, 2003.

[7] https://github.com/alecjacobson/gptoolbox (visited: 2019. 06. 03.)

THANK YOU FOR YOUR ATTENTION!



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