Optimization in interpolation by generalized periodic cubic splines

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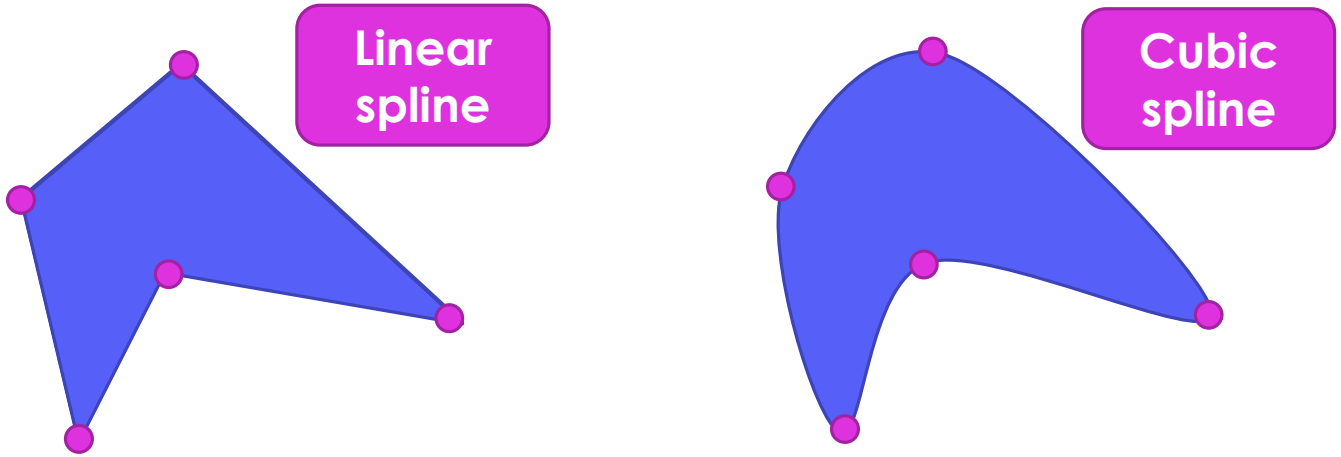
Motivation

- Interpolation problem for closed loops.



- We are looking for a continuously differentiable curve (how many times?).
- Periodic cubic spline interpolation.

Splines

- Spline functions are piecewise polynomials, smoothly connected to each other.

$$a, b \in \mathbb{R}, \quad a < b, \quad n \in \mathbb{N}, \quad n \geq 2$$

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

→ **Knots**

$$\mathcal{P}_\ell := \left\{ \sum_{k=0}^{\ell} a_k t^k \;\middle|\; t \in \mathbb{R},\ a_k \in \mathbb{R} \quad (k = 0, \ldots, \ell) \right\}$$

$$\mathcal{S}_\ell := \left\{ s \in C^{\ell-1}[a,b] \;\middle|\; s|_{[t_i, t_{i+1})} \in \mathcal{P}_\ell \right\}$$

$$s \in C^{\ell-1}[a,b] \iff s \in C^{\ell-1}\{t_i\} \iff s^{(k)}(t_i - 0) = s^{(k)}(t_i + 0)$$

$$(i = 1, \ldots, n-1) \qquad (k = 0, \ldots, \ell - 1)$$

- Problem: $f \in C(\mathbb{R}),\ f(x + (b-a)) = f(x) \quad (x \in \mathbb{R}),\ f_i := f(t_i)$
- Find: $s \in \mathcal{S}_3 \ :\ s(t_i) = f_i,\ s'(a+0) = s'(b-0),\ s''(a+0) = s''(b-0)$

Periodic cubic spline interpolation

- A second derivative of the cubic spline is a linear spline (linear between two adjacent knot points).
- We can construct a linear equation system for the second derivative values at the knots.

$$\begin{bmatrix} \frac{\Delta_0+\Delta_1}{3} & \frac{\Delta_1}{6} & 0 & 0 & \dots & 0 & \frac{\Delta_0}{6} \\ \frac{\Delta_1}{6} & \frac{\Delta_1+\Delta_2}{3} & \frac{\Delta_2}{6} & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \frac{\Delta_{n-2}}{6} & \frac{\Delta_{n-2}+\Delta_{n-1}}{3} & \frac{\Delta_{n-1}}{6} \\ \frac{\Delta_0}{6} & 0 & \dots & \dots & 0 & \frac{\Delta_{n-1}}{6} & \frac{\Delta_{n-1}+\Delta_0}{3} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \Delta_1^f - \Delta_0^f \\ \Delta_2^f - \Delta_1^f \\ \vdots \\ \vdots \\ \vdots \\ \Delta_{n-2}^f - \Delta_{n-1}^f \\ \Delta_{n-1}^f - \Delta_0^f \end{bmatrix}$$

- The solution exists and **unique**.

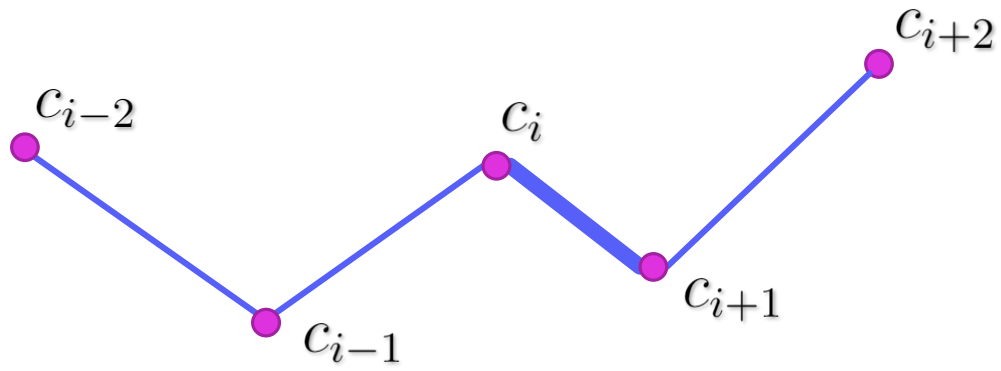
Cubic Spline Construction I.

$$\Delta_i := t_{i+1} - t_i, \qquad \Delta_i^f := \frac{f_{i+1} - f_i}{\Delta_i}$$

$$s|_{[t_i,t_{i+1})} \in \mathcal{P}_3 \implies s''|_{[t_i,t_{i+1})} \in \mathcal{P}_1$$

$$c_n = c_0,\ c_1,\ \ldots,\ c_{n-1},\ c_n$$

$$s''(t) = \frac{t - t_i}{\Delta_i} \cdot c_{i+1} + \frac{t_{i+1} - t}{\Delta_i} \cdot c_i \qquad \big(t \in [t_i, t_{i+1})\big)$$



Cubic Spline Construction II.

$$s''(t) = \frac{t - t_i}{\Delta_i} \cdot c_{i+1} + \frac{t_{i+1} - t}{\Delta_i} \cdot c_i$$

$\int$ ⬇

$$s'(t) = b_i + \frac{(t - t_i)^2}{2\Delta_i} \cdot c_{i+1} - \frac{(t_{i+1} - t)^2}{2\Delta_i} \cdot c_i$$

$\int$ ⬇

$$s(t) = a_i + b_i(t - t_i) + \frac{(t - t_i)^3}{6\Delta_i} \cdot c_{i+1} + \frac{(t_{i+1} - t)^3}{6\Delta_i} \cdot c_i$$

$$s \in C[a, b],\ s(t_i) = f_i \iff \begin{aligned} s(t_i + 0) &= f_i \Rightarrow a_i = f_i - \frac{\Delta_i^2}{6} c_i \\ &\Downarrow \\ s(t_{i+1} - 0) &= f_{i+1} \Rightarrow b_i = \Delta_i^f + \frac{\Delta_i}{6}(c_i - c_{i+1}) \end{aligned}$$

Cubic Spline Construction III.

$$s \in C^1[a,b] \iff s'(t_i+0) = s'(t_i-0)$$

$$s'(t_i-0) = b_{i-1} + \frac{\Delta_{i-1}}{2}c_i =$$

$$\Delta^f_{i-1} + \frac{\Delta_{i-1}}{6}(c_{i-1}-c_i) + \frac{\Delta_{i-1}}{2}c_i =$$

$$\Delta^f_{i-1} + \frac{\Delta_{i-1}}{6}c_{i-1} + \frac{\Delta_{i-1}}{3}c_i$$

$$s'(t_i+0) = b_i - \frac{\Delta_i}{2}c_i =$$

$$\Delta^f_i + \frac{\Delta_i}{6}(c_i - c_{i+1}) - \frac{\Delta_i}{2}c_i =$$

$$\Delta^f_i - \frac{\Delta_i}{6}c_{i+1} - \frac{\Delta_i}{3}c_i$$

Cubic Spline Construction IV.

$$\Delta_i^f - \frac{\Delta_i}{6}c_{i+1} - \frac{\Delta_i}{3}c_i = \Delta_{i-1}^f + \frac{\Delta_{i-1}}{6}c_{i-1} + \frac{\Delta_{i-1}}{3}c_i \quad \Longleftrightarrow$$

$$\Delta_i^f - \Delta_{i-1}^f = \frac{\Delta_{i-1}}{6}c_{i-1} + \frac{\Delta_{i-1}+\Delta_i}{3}c_i + \frac{\Delta_i}{6}c_{i+1}$$

$$\begin{bmatrix} \frac{\Delta_0+\Delta_1}{3} & \frac{\Delta_1}{6} & 0 & 0 & \dots & 0 & \frac{\Delta_0}{6} \\ \frac{\Delta_1}{6} & \frac{\Delta_1+\Delta_2}{3} & \frac{\Delta_2}{6} & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \frac{\Delta_{n-2}}{6} & \frac{\Delta_{n-2}+\Delta_{n-1}}{3} & \frac{\Delta_{n-1}}{6} \\ \frac{\Delta_0}{6} & 0 & \dots & \dots & 0 & \frac{\Delta_{n-1}}{6} & \frac{\Delta_{n-1}+\Delta_0}{3} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \Delta_1^f - \Delta_0^f \\ \Delta_2^f - \Delta_1^f \\ \vdots \\ \vdots \\ \vdots \\ \Delta_{n-2}^f - \Delta_{n-1}^f \\ \Delta_{n-1}^f - \Delta_0^f \end{bmatrix} \rightarrow Ac = d$$

SDD Matrix | Unique solution | Tridiagonal matrix+ dyad | Sherman-Morrison formula | Thomas' algorithm (O(n))

Cubic Spline (Summary)

$\Delta_i \rightarrow A$

$\Delta_i^f \rightarrow d$

$\rightarrow Ac = d$

$\downarrow$

c

$\downarrow$

$$a_i = f_i - \frac{\Delta_i^2}{6} c_i$$

$$b_i = \Delta_i^f + \frac{\Delta_i}{6}(c_i - c_{i+1})$$

$$s(t) = a_i + b_i(t - t_i) + \frac{(t - t_i)^3}{6\Delta_i} \cdot c_{i+1} + \frac{(t_{i+1} - t)^3}{6\Delta_i} \cdot c_i$$

Problem

- We need not only one solution, but an infinite set of solutions (shape factor, optimization).
- Continuously differentiability is enough.
- Interpolation conditions should not be violated.
- Solutions should to be periodic, and piecewise polynomial.

$$\mathcal{S}^* := \left\{ s \in C^1(\mathbb{R}) \quad \middle| \quad s|_{[t_i,t_{i+1})} \in \mathcal{P}_3,\ s(t+(b-a)) = s(t),\ t \in \mathbb{R} \right\}$$

- These functions are not (periodic cubic) splines.

$$s \in C^2([a,b] \setminus \{t_1, \ldots, t_n\})$$

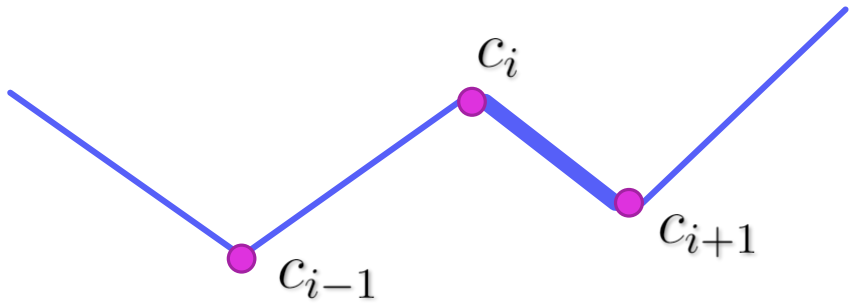
Almost…

Generalized cubic spline construction I.

$$s|_{[t_i,t_{i+1})} \in \mathcal{P}_3 \implies s''|_{[t_i,t_{i+1})} \in \mathcal{P}_1$$

$$c_n = c_0,\ c_1,\ \ldots,\ c_{n-1},\ c_n$$

$$s''(t) = \frac{t - t_i}{\Delta_i} \cdot c_{i+1} + \frac{t_{i+1} - t}{\Delta_i} \cdot c_i \qquad \big(t \in [t_i, t_{i+1})\big)$$

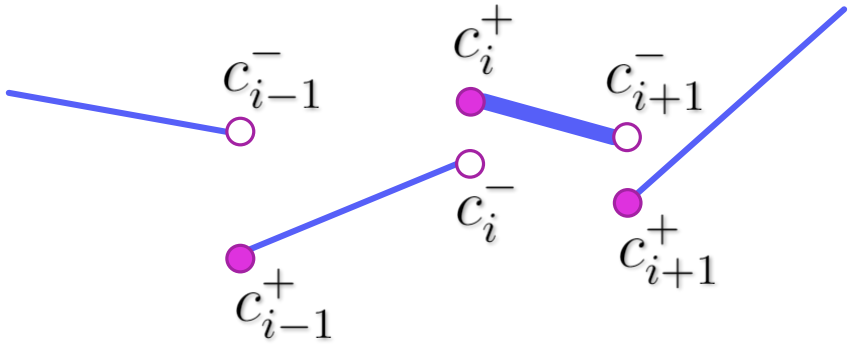


Generalized cubic spline construction II.

$$s|_{[t_i,t_{i+1})} \in \mathcal{P}_3 \implies s''|_{[t_i,t_{i+1})} \in \mathcal{P}_1$$

$$c_n = c_0,\ c_1,\ \ldots,\ c_{n-1},\ c_n$$

$$s''(t) = \frac{t - t_i}{\Delta_i} \cdot c_{i+1}^- + \frac{t_{i+1} - t}{\Delta_i} \cdot c_i^+ \qquad \big(t \in [t_i, t_{i+1})\big)$$



Linear equation system?

- Our formulas are almost exactly the same…

$$a_i = f_i - \frac{\Delta_i^2}{6} c_i^+$$

$$b_i = \Delta_i^f + \frac{\Delta_i}{6}(c_i^+ - c_{i+1}^-)$$

$$s(t) = a_i + b_i(t - t_i) + \frac{(t - t_i)^3}{6\Delta_i} \cdot c_{i+1}^- + \frac{(t_{i+1} - t)^3}{6\Delta_i} \cdot c_i^+$$

… but

$$\Delta_i^f - \Delta_{i-1}^f = \frac{\Delta_{i-1}}{6} c_{i-1}^{\oplus} + \frac{\Delta_i}{3} c_i^{\oplus} + \frac{\Delta_i}{6} c_{i+1}^{\ominus} + \frac{\Delta_{i-1}}{3} c_i^{\ominus}$$

- For example, if the LHS derivatives are known, the RHS derivatives can be calculated.

Linear equation system

$$A^+ := \begin{bmatrix} \frac{\Delta_1}{3} & 0 & 0 & 0 & \dots & 0 & \frac{\Delta_0}{6} \\ \frac{\Delta_1}{6} & \frac{\Delta_2}{3} & 0 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \frac{\Delta_{n-2}}{6} & \frac{\Delta_{n-1}}{3} & 0 \\ 0 & 0 & \dots & \dots & 0 & \frac{\Delta_{n-1}}{6} & \frac{\Delta_0}{3} \end{bmatrix} \qquad c^+ = \begin{bmatrix} c_1^+ \\ c_2^+ \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1}^+ \\ c_n^+ \end{bmatrix} \qquad d := \begin{bmatrix} \Delta_1^f - \Delta_0^f \\ \Delta_2^f - \Delta_1^f \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \Delta_{n-2}^f - \Delta_{n-1}^f \\ \Delta_{n-1}^f - \Delta_0^f \end{bmatrix}$$

$$A^- := \begin{bmatrix} \frac{\Delta_0}{3} & \frac{\Delta_1}{6} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{\Delta_1}{3} & \frac{\Delta_2}{6} & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{\Delta_{n-2}}{3} & \frac{\Delta_{n-1}}{6} \\ \frac{\Delta_0}{6} & 0 & \dots & \dots & 0 & 0 & \frac{\Delta_{n-1}}{3} \end{bmatrix} \qquad c^- = \begin{bmatrix} c_1^- \\ c_2^- \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1}^- \\ c_n^- \end{bmatrix}$$

$$A^+c^+ + A^-c^- = d$$

$$A^+c^+ = d - A^-c^-$$

$$A = A^+ + A^-$$

Generalized cubic spline construction process

$\Delta_i \rightarrow A^+, A^-$

$\Delta_i^f \rightarrow d$

$\xrightarrow{c^-}$ $A^+c^+ = d - A^-c^-$

LHS 2nd derivatives are parameters

c^+

$$a_i = f_i - \frac{\Delta_i^2}{6}c_i^+$$

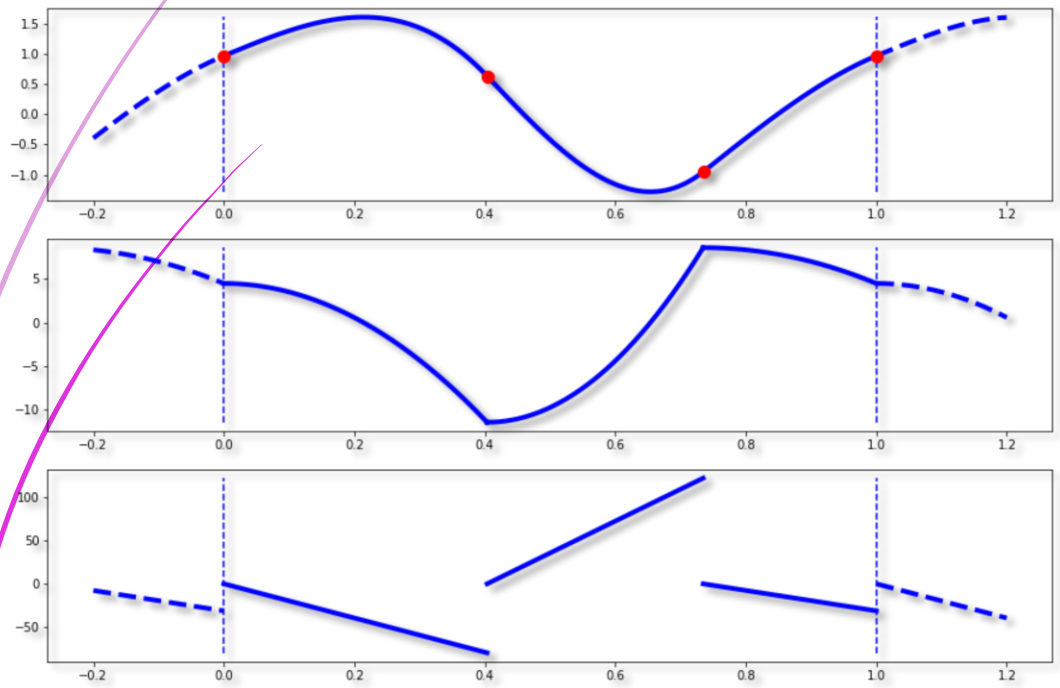
$$b_i = \Delta_i^f + \frac{\Delta_i}{6}(c_i^+ - c_{i+1}^-)$$

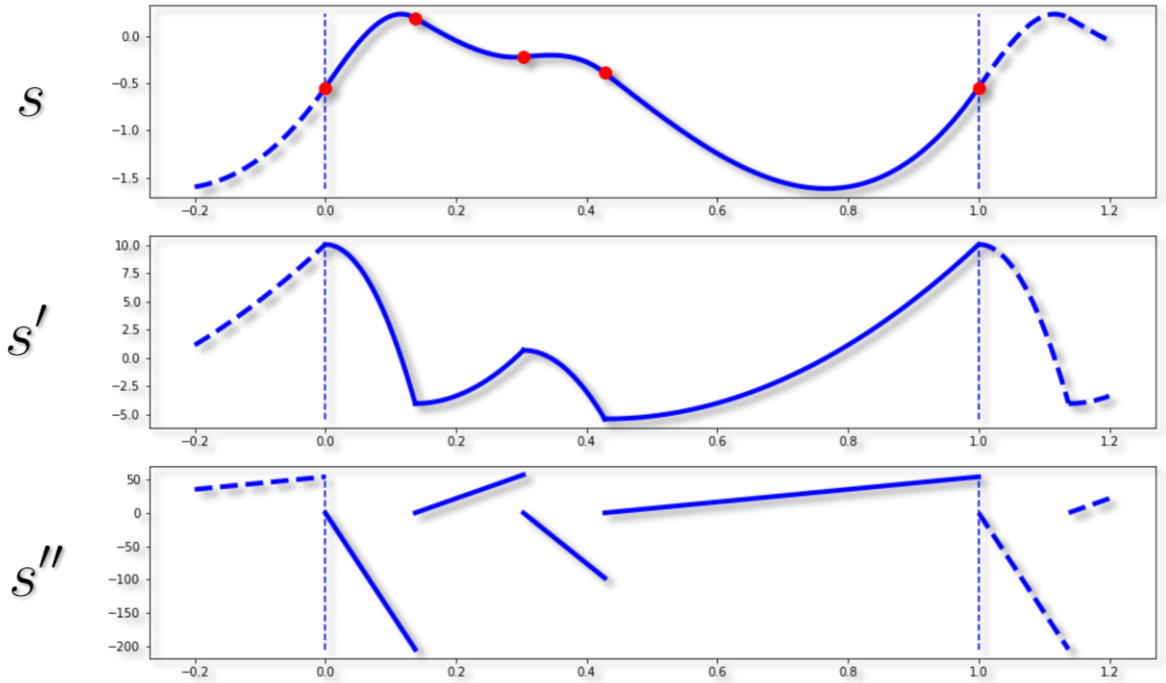
The spline is a function of LHS 2nd derivative vector

$$s(t) = a_i + b_i(t - t_i) + \frac{(t - t_i)^3}{6\Delta_i} \cdot c_{i+1}^- + \frac{(t_{i+1} - t)^3}{6\Delta_i} \cdot c_i^+$$

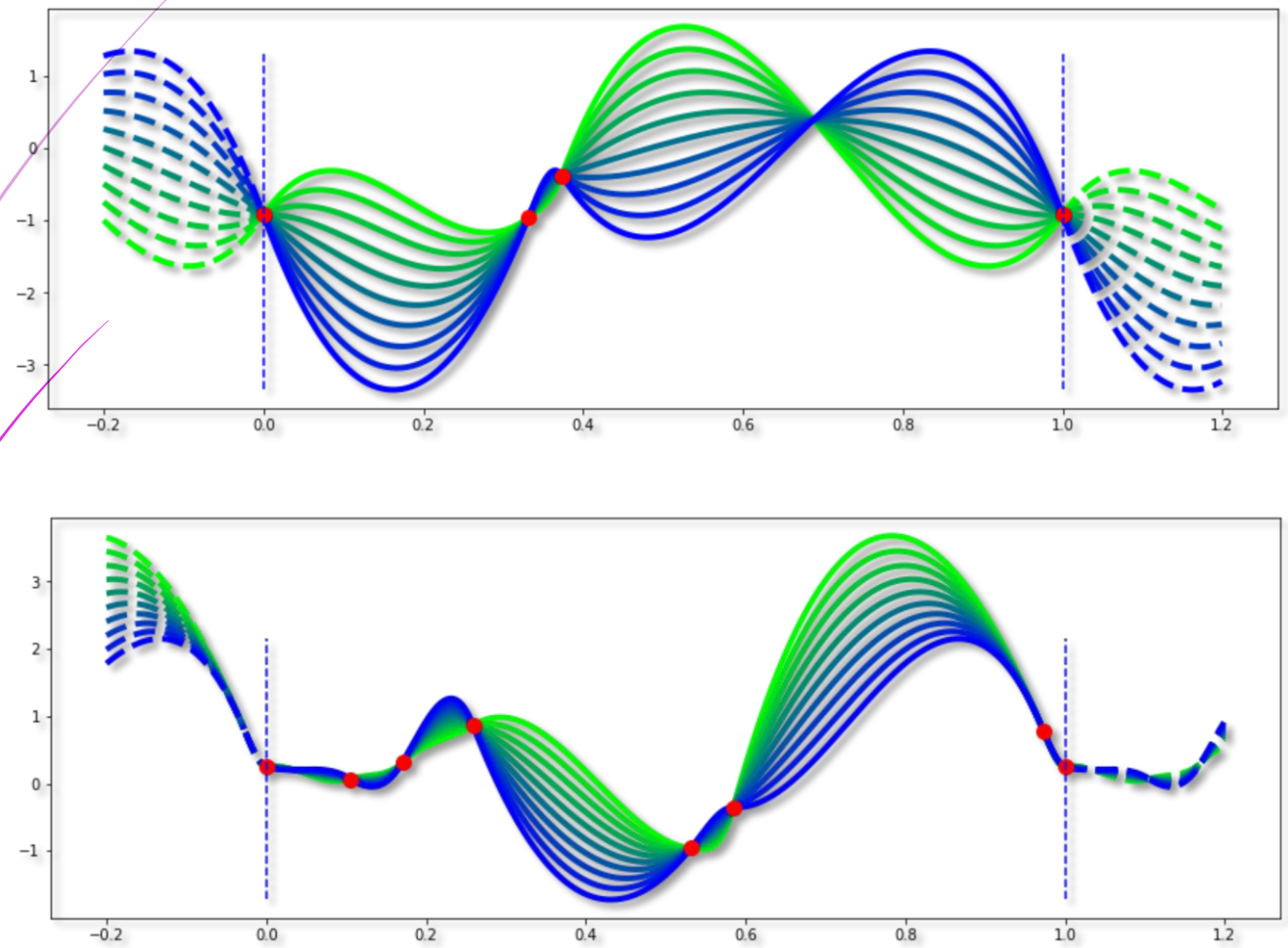
Examples in 1D

- Let's try our method!





Infinitely many solutions

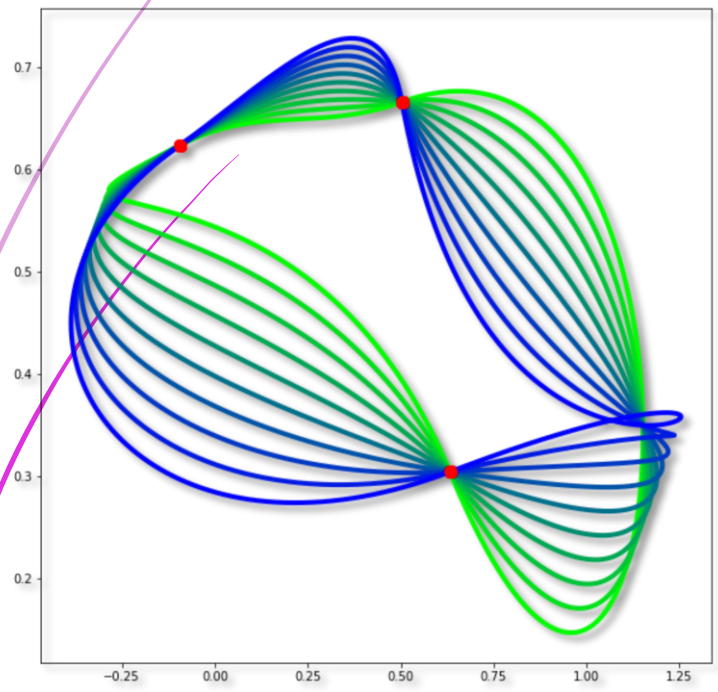


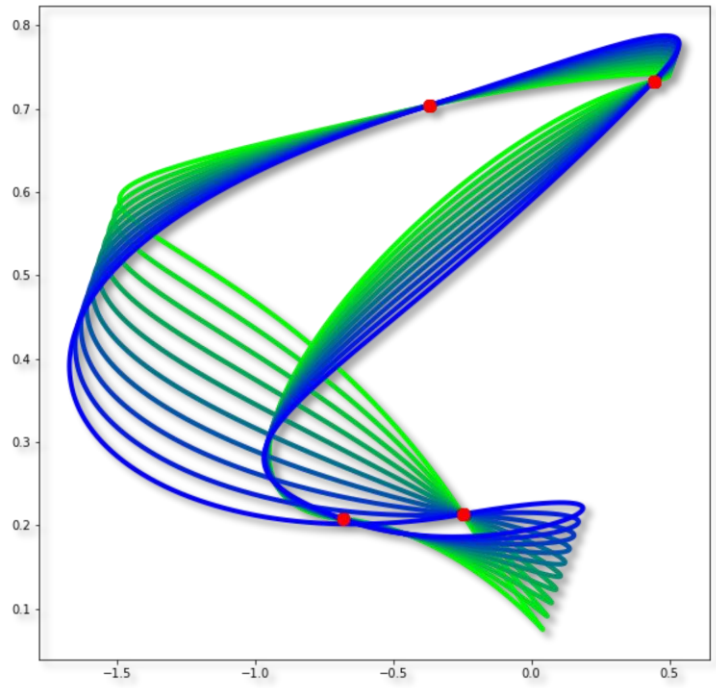
Different LHS 2nd derivative vectors

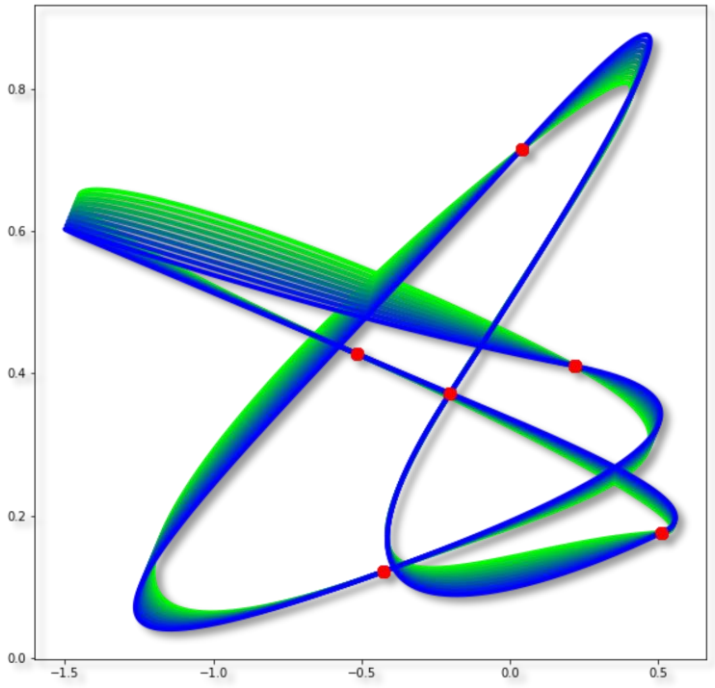
Different solutions from S*

Examples in 2D

- The process can be naturally generalized to 2D.







- Nodes should to be visited in a given order.

Optimization problem

- Objective function and optimization problem:

$$\Phi : \mathcal{S}^* \to \mathbb{R}$$

$$s^* = \underset{s \in \mathcal{S}^*}{\operatorname{argmin}}\, \Phi(s)$$

This is a function of LHS 2nd derivative vector

- We can minimize…
  - …the approximate total curvature:

$$\Phi(s) := \int_a^b \left(s''(t)\right)^2 \, \mathrm{d}t$$

  - …the squared error:

$$\Phi(s) := \int_a^b \left(f(t) - s(t)\right)^2 \, \mathrm{d}t$$

  - …and much more.

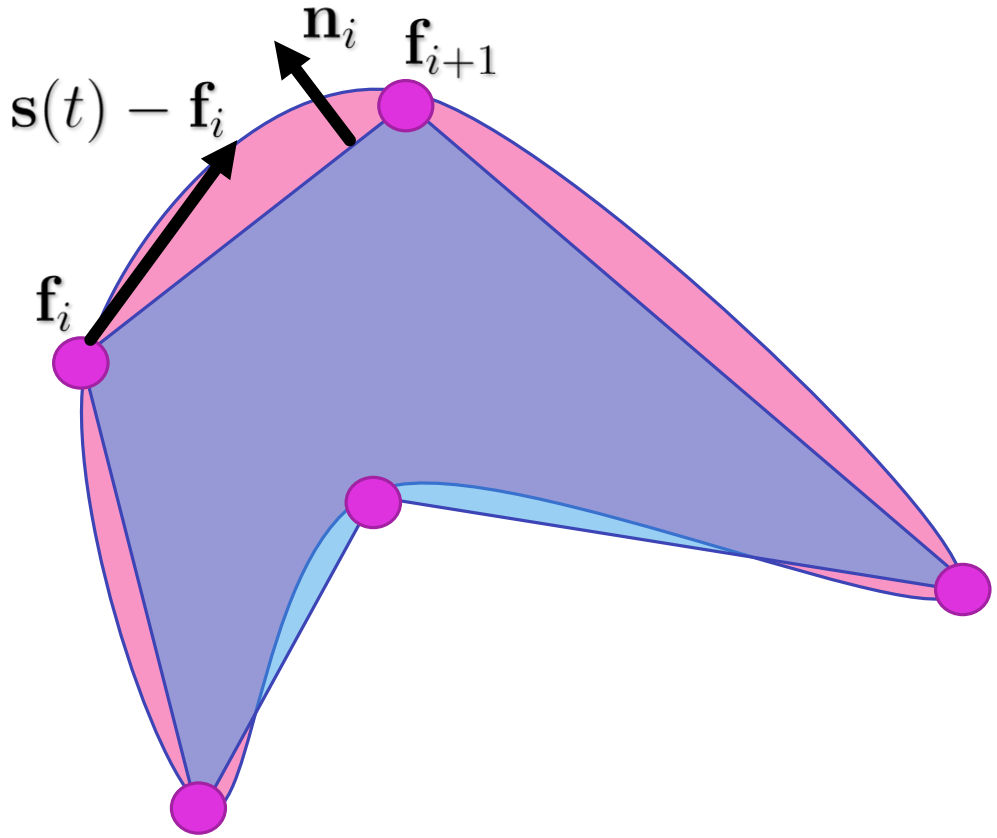
Optimization problem in 2D

$$(t_i, \mathbf{f}_i) \implies \mathbf{s}(t)$$

$$\mathbf{n}_i \perp (\mathbf{f}_{i+1} - \mathbf{f}_i), \ \|\mathbf{n}\|_2 = 1$$

$$\Phi(\mathbf{s}) := \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \mathbf{s}(t) - \mathbf{f}_i, \ \mathbf{n}_i \rangle^2 \ \mathrm{d}t$$

- Squared distance of the spline and a polygon line!



On calculating the minimizer I.

- Consider a quadratic objective function, e.g.:

$$\Phi(s) := \int_a^b \left(s''(t)\right)^2 \, \mathrm{d}t$$

- The objective function is quadratic function of LHS 2nd derivatives!
- It is quite easy to solve the optimization problem (numerically).
- Is it also easy to find the exact solution?

On calculating the minimizer II.

$$\int_a^b \left(s''(t)\right)^2 \,\mathrm{d}t = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\frac{t-t_i}{\Delta_i} \cdot c_{i+1}^- + \frac{t_{i+1}-t}{\Delta_i} \cdot c_i^+ \right)^2 \,\mathrm{d}t =$$

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\frac{t-t_i}{\Delta_i}\right)^2 (c_{i+1}^-)^2 + 2\frac{(t-t_i)(t_{i+1}-t)}{\Delta_i^2} c_{i+1}^- c_i^+ + \left(\frac{t_{i+1}-t}{\Delta_i}\right)^2 (c_i^+)^2 \,\mathrm{d}t =$$

$$\int_{t_i}^{t_{i+1}} (t-t_i)^2 \,\mathrm{d}t = \frac{\Delta_i^3}{3}$$

$$\int_{t_i}^{t_{i+1}} (t-t_i)(t_{i+1}-t) \,\mathrm{d}t = \frac{\Delta_i^3}{6}$$

$$\int_{t_i}^{t_{i+1}} (t_{i+1}-t)^2 \,\mathrm{d}t = \frac{\Delta_i^3}{3}$$

$$\sum_{i=0}^{n-1} \frac{\Delta_i}{3} \left[(c_{i+1}^-)^2 + c_{i+1}^- c_i^+ + (c_i^+)^2 \right]$$

On calculating the minimizer III.

$$\sum_{i=0}^{n-1} \frac{\Delta_i}{3}\left[(c_{i+1}^-)^2 + c_{i+1}^- c_i^+ + (c_i^+)^2\right]$$

Quadratic function of the LHS 2nd derivatives

$$A^+c^+ = d - A^-c^-$$

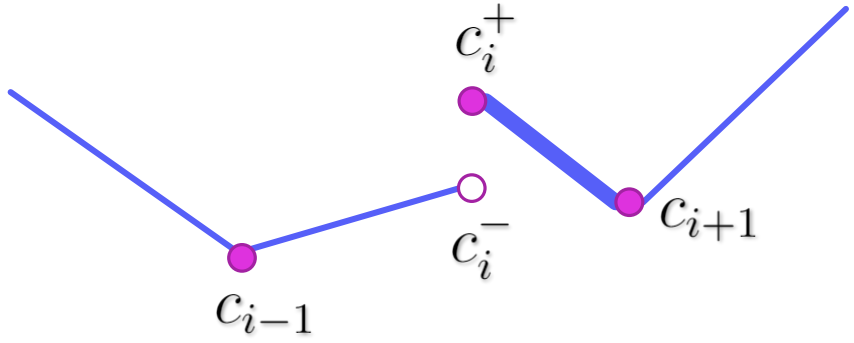
$$c^+ = (A^+)^{-1}d - (A^+)^{-1}A^-c^-$$

Linear function of the LHS 2nd derivatives

- Quadratic objective functions produce quadratic forms to be minimized.
- The minimizer theoretically can be determined explicitly.
- Expressions may easily become complicated.
- Special cases can be handled.

Cubic spline interpolation revisited I.

- Twice continuous differentiability in knots is retrievable.



$$A^{\pm} = \left[a_1^{\pm},\ a_2^{\pm}, \ldots, a_n^{\pm}\right]$$

$$J \subset \{1, 2, \ldots, n\}$$

Indices of C2 knots

$$A' = \left[a'_1, a'_2, \ldots, a'_n\right] \qquad a'_j = \begin{cases} a_j^+ + a_j^- & j \in J \\ a_j^- & j \notin J \end{cases}$$

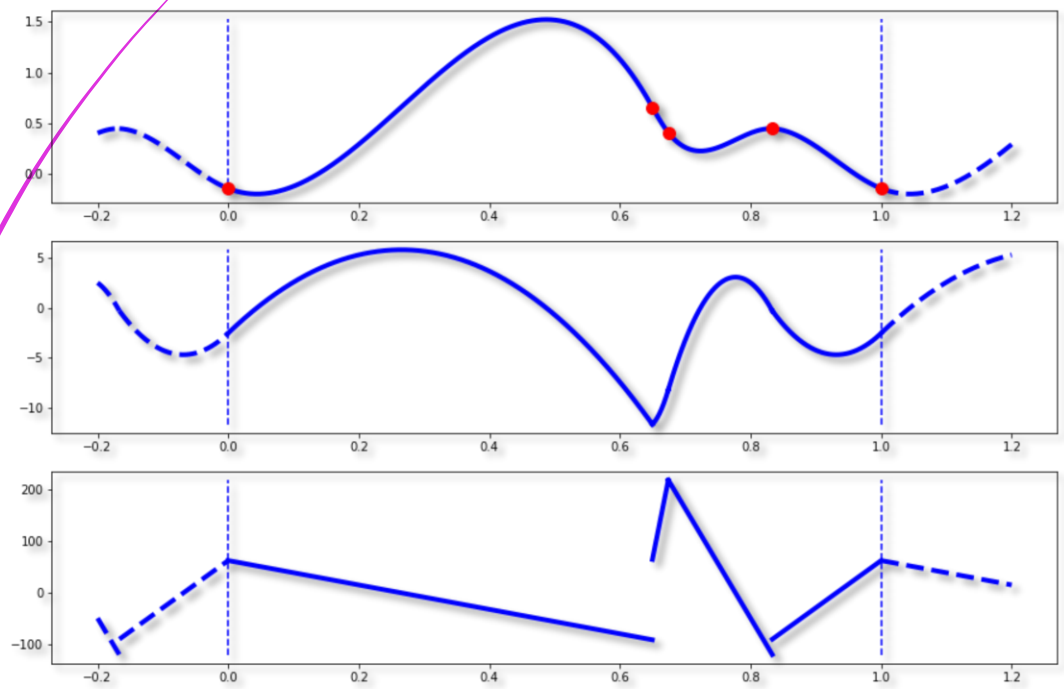
$$A'' = \left[a''_1, a''_2, \ldots, a''_n\right] \qquad a''_j = \begin{cases} 0 & j \in J \\ a_j^+ & j \notin J \end{cases}$$

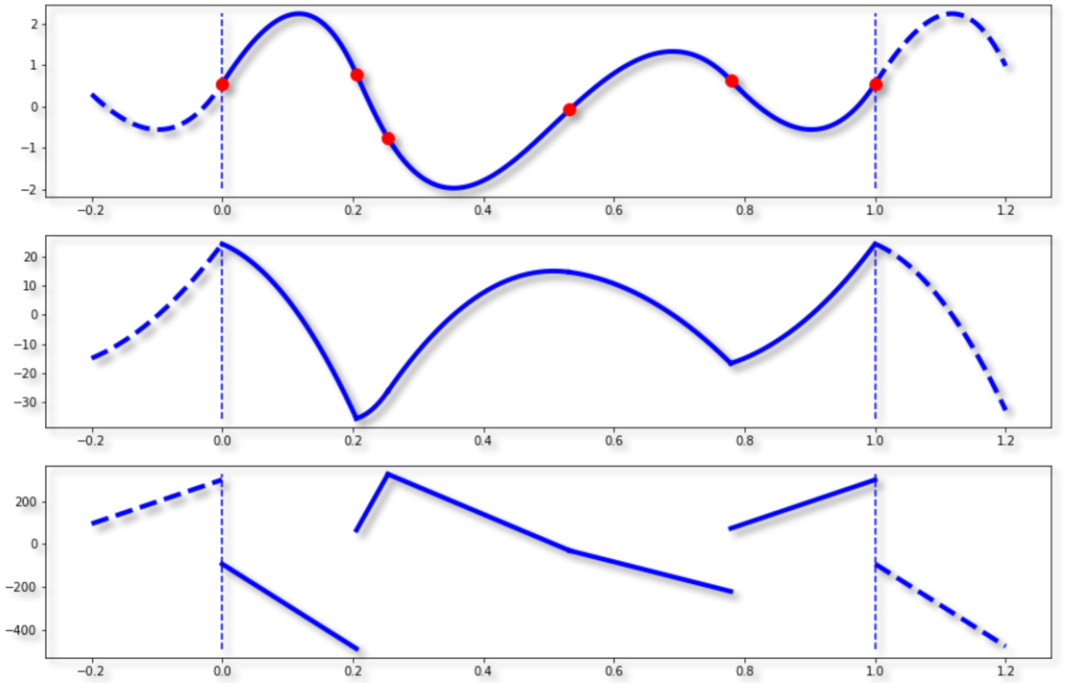
$$\Rightarrow \quad A''c'' = d - A'c'$$

Cubic spline interpolation revisited II.

$J = \emptyset$ → **Problem detailed before**

$J = \{1, 2, \ldots, n\}$ → **Standard periodic cubic spline interpolation**





Summary

- Optimization parameters can be obtained by violating continuity of the second derivative.
- Various optimization problems can be formulated.
- Minimizers of some important quadratic objective functions can be determined explicitly.
- Exact calculation of minimizers could be complex.
- If Thomas' algorithm is stable, we can solve the problem in linear time.

Thank you for you attention!