LINE SIMPLIFICATION WITH SEQUENCE OF BEST APPROXIMATIONS

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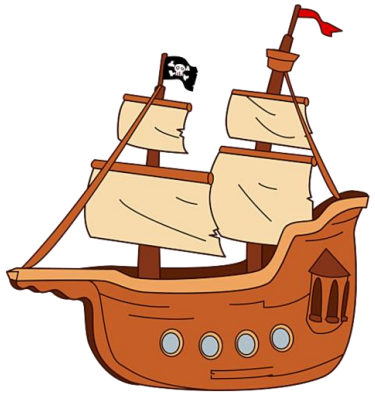
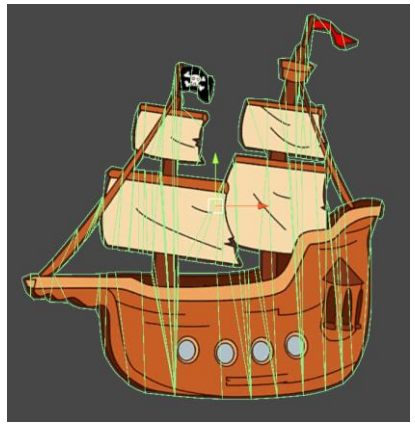
Problem

- Given a polyline P. Find a P' polyline that is a **good approximation** of P and it has significantly **fewer vertices**.
- It would be good if …
  - … the **distance** between P and P' could be somehow calculated.
  - … the **number of desired vertices** of P' could be set accurately.

- Application: **polygon collider** in 2D physics simulation.





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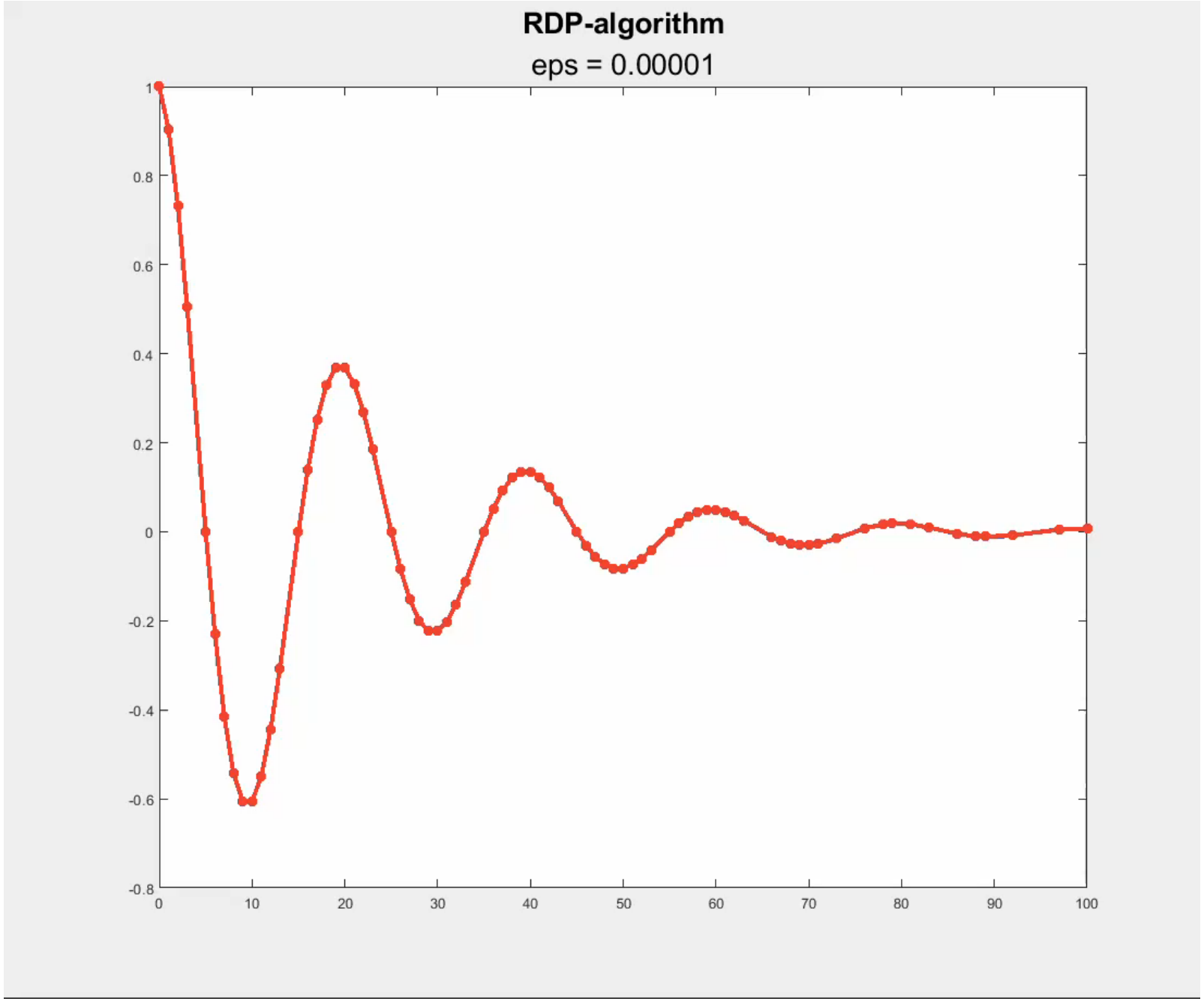

Ramer-Douglas-Peucker Algorithm

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Source: Jade Haayen – Douglas Peucker Animated for presentation.
https://www.youtube.com/watch?v=TvHmIWM_EUU

- Advantages
  - Easy implementation.
  - Easy generalization to 2D.
  - Good approximation property for many points.
- Disadvantages
  - We do not get error, only a tolerance parameter.
  - Number of points can not be controlled.
  - Not so good approximation for few points

- Our approach: approximation in ~~Hilbert-space~~ increasing subspaces of a Hilbert-space.

RDP Algorithm Overview

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- Finite dimensional Hilbert-space

$$(H, \langle \cdot, \cdot \rangle)$$

- Norm induced by inner product

$$f \in H$$

$$\|f\| := \sqrt{\langle f, f \rangle}$$

- Basis of sequence of subspaces

$$b_1, b_2, \ldots, b_k, b_{k+1} \ldots \in H$$

$$H_k := \operatorname{span}(b_1, \ldots, b_k)$$

$$\dim H_k = k$$

$$H_k \subset H_{k+1}$$

Sequence of increasing subspaces of Hilbert-space

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- Gram-Schmidt Orthonormalization

$$q_1 = b_1 \qquad e_1 = \frac{q_1}{\|q_1\|}$$

$$q_k = b_k - \sum_{j=1}^{k-1} \langle b_k, e_j \rangle e_j \qquad e_k = \frac{q_k}{\|q_k\|}$$

- Best approximation in subspace

$$f_k := \sum_{j=1}^{k} \langle f, e_j \rangle e_j$$

- Distance from subspace

$$d_k := \|f - f_k\|$$

Best Approximation in subspace

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- **Lemma 1.**

$$d_{k+1}^2 = d_k^2 - \langle f, e_{k+1} \rangle^2$$

- **Lemma 2.**

$$\langle f, e_{k+1} \rangle^2 = \frac{\langle b_{k+1},\ f - f_k \rangle^2}{\|b_{k+1}\|^2 - \sum_{j=1}^{k} \langle b_{k+1}, e_j \rangle^2}$$

Distance change in approximation sequence

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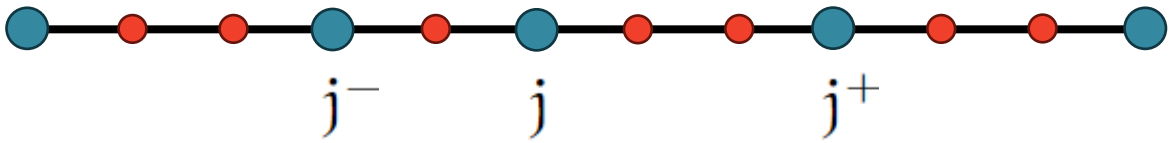
- Subdivision of an interval (knots)

$$[0, n]$$
$$J_n := \{0, 1, \ldots, n\}$$

$$J_m := \{j_1, \ldots, j_m\} \subset J_n$$
$$\{0, n\} \subset J_m$$
$$\forall i, k = 1, \ldots, m \ : \ i \neq k \implies j_i \neq j_k$$

- Previous/next knot points

$$j^- := \max\{j' \in J_m \ : \ j' < j\}$$
$$j^+ := \min\{j' \in J_m \ : \ j' > j\}$$



Piecewise Linear Functions 1/3

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Piecewise Linear Functions 2/3

- Piecewise linear periodic functions

$$\mathrm{PL}_{\mathrm{per}}(J_m, \mathbb{R})$$

$$f_{|[j,j^+]}(x) = f(j) + \underbrace{\frac{f(j^+) - f(j)}{j^+ - j}}_{\Delta f(j)}(x - j) =$$

$$= f(j) + \Delta f(j)(x - j)$$

Piecewise Linear

$$f(0) = f(n)$$

Periodic

- Linear space, subspace of $C([0, n])$
- $\dim \mathrm{PL}_{\mathrm{per}}(J_m, \mathbb{R}) = |J_m| - 1$
- If $J_m \subset J_k$ then $\mathrm{PL}_{\mathrm{per}}(J_m, \mathbb{R})$ is subspace of $\mathrm{PL}_{\mathrm{per}}(J_k, \mathbb{R})$
- $\left(\mathrm{PL}_{\mathrm{per}}(J_m, \mathbb{R}), \langle \cdot, \cdot \rangle\right)$ is finite dimensional Hilbert-space

w.r.t. $\langle f, g \rangle := \int_0^n f(x)g(x)dx$

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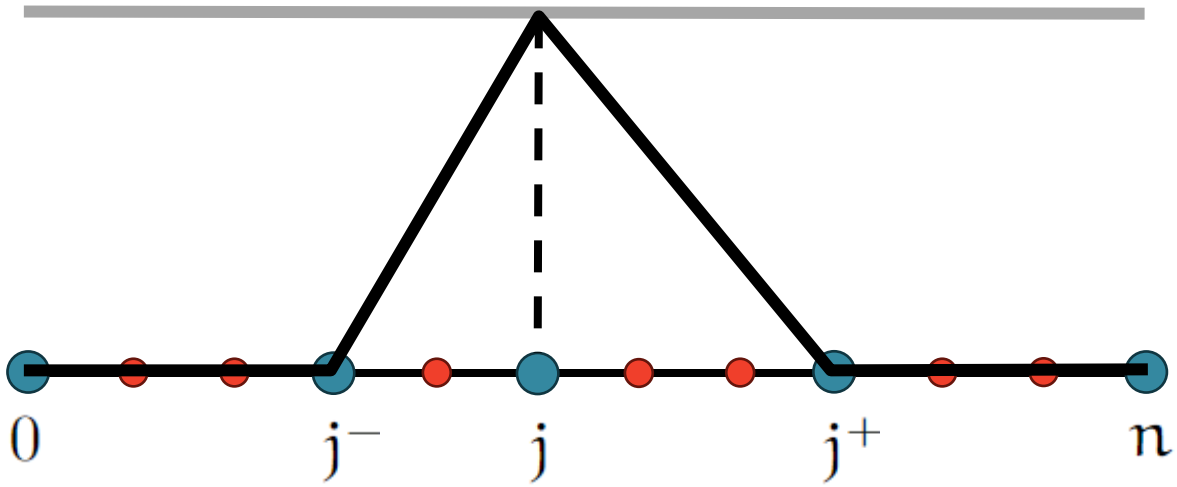
- Basis functions in $\mathrm{PL}_{\mathrm{per}}(J_m, \mathbb{R})$
- Constant function + linear B-splines

$$B_j^J(x) := \begin{cases} \dfrac{x - j^-}{j - j^-} & x \in [j^-, j] \\ \dfrac{j^+ - x}{j^+ - j} & x \in [j, j^+] \end{cases}$$



Piecewise Linear Functions 3/3

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- Calculating the inner product

$$\int_0^n f(x)g(x)dx = \sum_{j\in J} \int_j^{j^+} f(x)g(x)dx$$

- On one subinterval

$$\int_j^{j^+} f(x)g(x)dx = \frac{j^+ - j}{6}\left[f(j)g(j) + f(j^+)g(j^+) + \right.$$
$$\left. + 4 \cdot \frac{f(j) + f(j^+)}{2} \cdot \frac{g(j) + g(j^+)}{2}\right] =$$
$$= \frac{j^+ - j}{6} \cdot \left[2\big(f(j)g(j) + f(j^+)g(j^+)\big) + \right.$$
$$\left. + f(j^+)g(j) + f(j)g(j^+)\right]$$

Calculating the inner product 1/3

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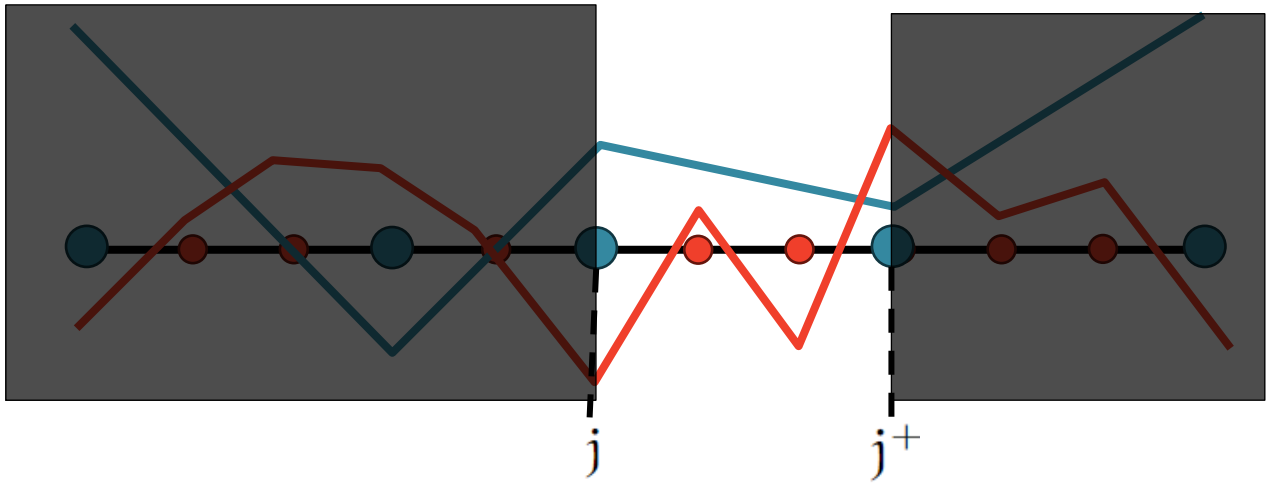
- How many operations should we use?

$$f \in \mathrm{PL}_{\mathrm{per}}(J_n, \mathbb{R}) \qquad g \in \mathrm{PL}_{\mathrm{per}}(J_m, \mathbb{R}) \qquad m \ll n$$



- Cost of calculating $\langle f, g \rangle$: $\mathcal{O}(n)$
- Is it possible to calculate the same in $\mathcal{O}(m)$?

Calculating the inner product 2/3

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- **Theorem 1.** Let us define the following $F, \Phi \in PL_{per}(J_n, \mathbb{R})$ functions:

$$F(i+1) = \sum_{k=0}^{i} \frac{f(k) + f(k+1)}{2}$$

$$\Phi(i+1) = \sum_{k=0}^{i} F(k) + \frac{f(k)}{3} + \frac{f(k+1)}{6}$$

Then

$$\int_j^{j^+} f(x)g(x)dx = F(j^+)g(j^+) - F(j)g(j) - \Delta g(j)\big(\Phi(j^+) - \Phi(j)\big)$$

where $j^+ := \min\{j' \in J_m \ : \ j' > j\}$.

- Calculation of F, Φ takes $\mathcal{O}(n)$ operations.
- After that $\langle f, g \rangle$ can be calculated in $\mathcal{O}(m)$!
- It is worth doing precomputation, if f occurs repeatedly in our calculations.

Calculating the inner product 3/3

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SequentialBestApproximation(f, m)

1: $J_1 := \{0, n\}$ // Initial knots
2: $e_1 := \frac{1}{\sqrt{n}}$ // First basis function
3: $f_1 := \langle f, e_1 \rangle e_1$ // Calculate approximation
4: $d_1^2 = \|f - f_1\|^2$ // Calculate squared distance
5: **for all** $k \in 1, \ldots, m-1$ **do**
6: $j_{k+1} :=$ ChooseNextIndex(. . .) // Choose next knot
7: $J_{k+1} := J_k \cup \{j_{k+1}\}$ // Update knots
8: $b_{k+1} := B_{j_{k+1}}^{J_{k+1}}$ // Create new basis
9: $q_{k+1} := b_{k+1} - \sum_{i=1}^{k} \langle b_{k+1}, e_i \rangle e_i$ // Orthogonalize
10: $e_{k+1} := q_{k+1} / \|q_{k+1}\|$ // Normalize
11: $f_{k+1} = f_k + \langle f, e_{k+1} \rangle e_{k+1}$ // Update approximation
12: $d_{k+1}^2 = d_k^2 - \langle f, e_{k+1} \rangle^2$ // Update squared distance
13: **end for**

SBA Algorithm 2/1

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ChooseNextIndex(…)

1: **for all** $j \in J_n \setminus J_k$ **do** // Try next knot
2: $\widetilde{J} := J_k \cup \{j\}$ // Update temporal knots
3: $\widetilde{b} := B_j^{\widetilde{J}}$ // Create temporal basis
4: $\widetilde{q} := \widetilde{b} - \sum_{i=1}^{k} \langle \widetilde{b}, e_i \rangle e_i$ // Orthogonalize
5: $\widetilde{e} := \widetilde{q} / \|\widetilde{q}\|$ // Normalize
6: $\widetilde{f} := f_k + \langle f, \widetilde{e} \rangle \widetilde{e}$ // Update approximation
7: $\widetilde{d}^2(j) := d_k^2 - \langle \widetilde{f}, \widetilde{e} \rangle^2$ // Update squared distance
8: **end for**
9: $J_{k+1} := \operatorname{argmin} \{ j \in J \setminus J_k \ : \ \widetilde{d}^2(j) \}$

- Trying all possible knots takes $\sim (n-k)k^2$ operations.

SBA Algorithm 2/1.5

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ChooseNextIndex(...)

1: **for all** $j \in J_n \setminus J_k$ **do** // Try next knot

2: $\widetilde{J} := J_k \cup \{j\}$ // Update temporal knots

3: $\widetilde{b} := B_j^{\widetilde{J}}$ // Create temporal basis

4: $\widetilde{g}(j) := \frac{\left\langle \widetilde{b},\ f-f_k \right\rangle^2}{\|\widetilde{b}\|^2 - \sum_{j=1}^{k} \left\langle \widetilde{b}, e_j \right\rangle^2}$ // Calculate expected improvement

5: **end for**

6: $j_{k+1} := \operatorname{argmax} \{ j \in J \setminus J_k \ : \ \widetilde{g}(j) \}$

- Trying all possible knots takes „only” $\sim (n-k)k$ operations.

- The SBA algorithm can be computed with $\mathcal{O}(nm^2)$ operations.
- RDP algorithm uses $\mathcal{O}(n^2)$ operations, but it can be speed up using special data sets.
- If $m \ll n$ our algorithm is quite efficient.

SBA Algorithm 2/2

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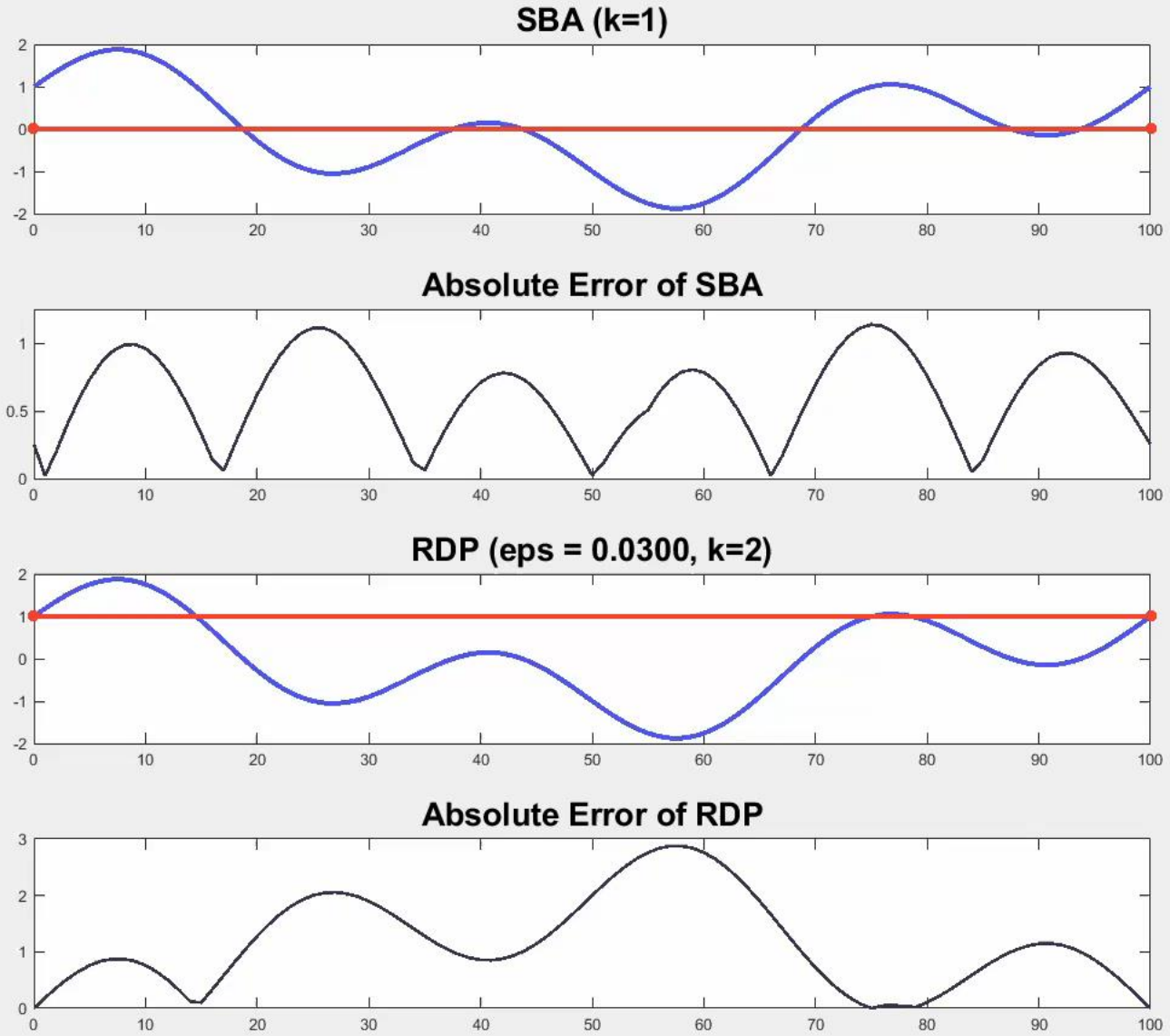

SBA vs. RDP

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- What if the range of f is n-dimensional?

$$f \in PL_{per}(J_n, \mathbb{R}^2)$$

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- The coordinate functions are approximated separately

$$x \approx x_k = \sum_{j=1}^{k} \langle f, e_{x,k} \rangle e_{x,k} \qquad y \approx y_k = \sum_{j=1}^{k} \langle f, e_{y,k} \rangle e_{y,k}$$

- The basis functions are from orthogonal subspaces for different dimensions, i.e.

$$\langle e_{x,i}, e_{y,j} \rangle = 0$$

- The improvement of the approximation

$$d_{k+1}^2 = d_k^2 - \langle x, e_{x,k+1} \rangle^2 - \langle y, e_{y,k+1} \rangle^2$$

Generalization to 2D

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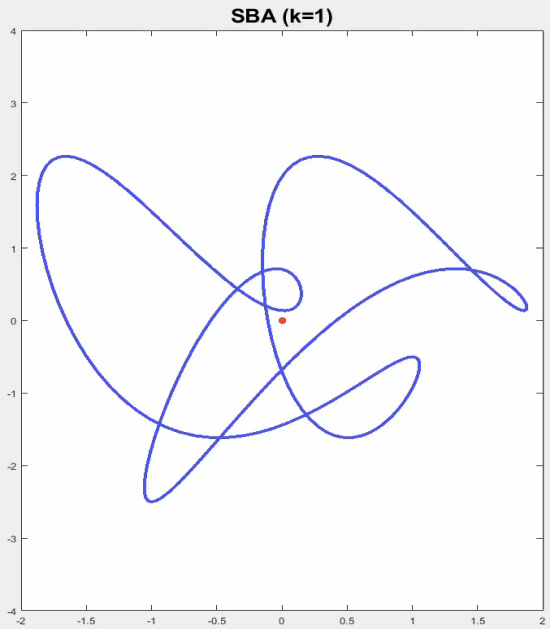
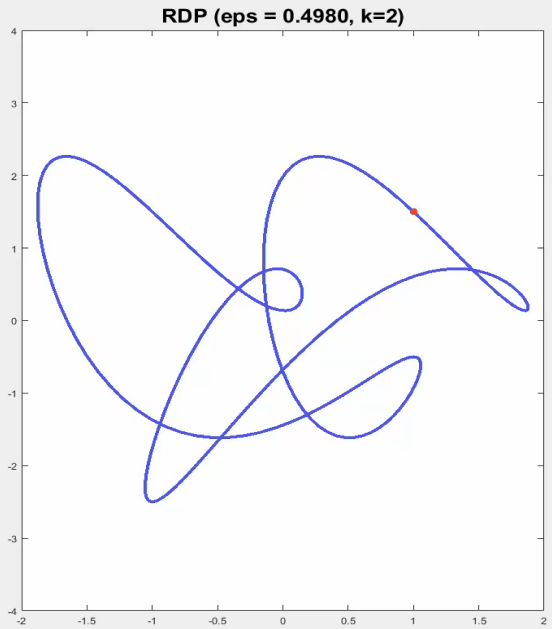
SBA in 2D

- Example for approximate a loop in 2D.
- Left: SBA algorithm.
- Right: MATLAB `reducepoly` function, which uses RDP algorithm for polygon simplification.





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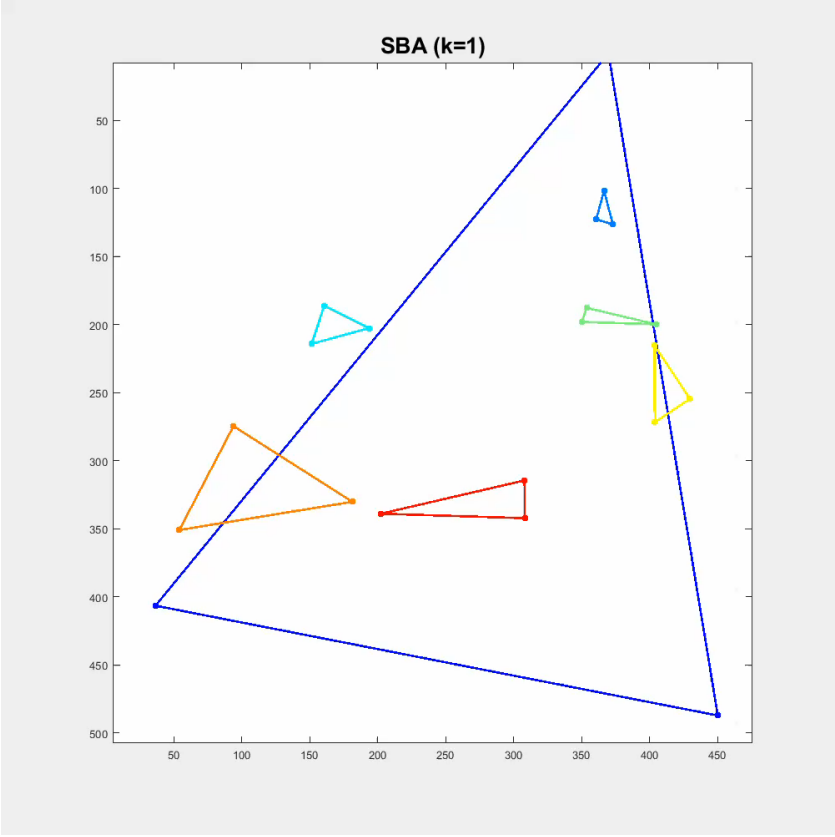
SBA for contours

- Example for approximate multiple loops in 2D
- In every step we improve the approximation of the loop which has the largest error.



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Future work

- Implementation in C#/Unity
- Better knot choosing strategy
- Generalization for (non-PL) parametric curves
- Generalization for surfaces

20 / 20

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THANK YOU FOR YOUR ATTENTION!