ON TOPOLOGICAL PROPERTIES OF POLYGONAL MESHES

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MOTIVATION

- Physical objects are often used in CG, but not properly defined.
- Widely accepted approach is based on CSG, not on B-rep.
- Strong requirements are not needed.
- Assuming 2-manifoldity is a good way (Half-edge, Winged-edge).
- Missing unified mathematical model.
- Most of statements are known, but yet not proven (missing model).

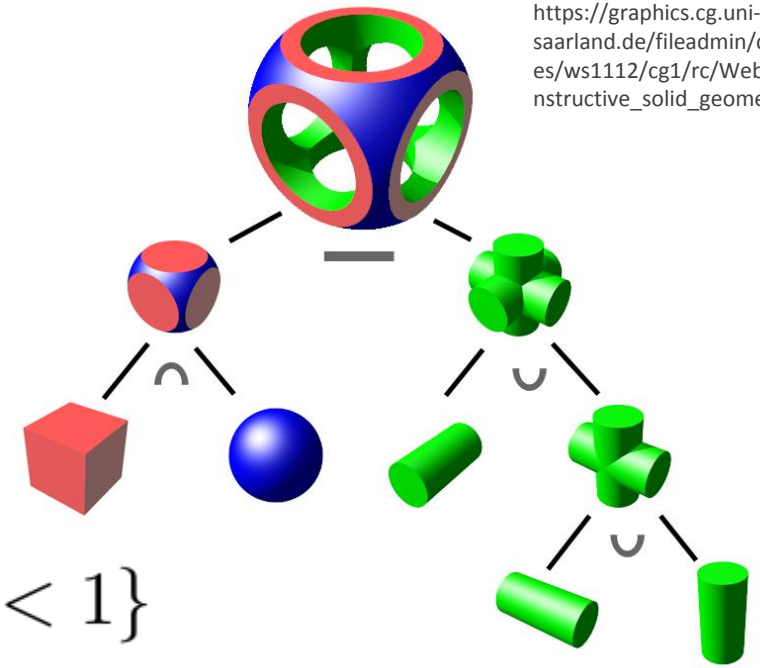
OVERVIEW

- First approach of solids
  - Ray Casting
  - Regular sets
- A new approach.
  - Boundary representation and possible errors
  - Well-defined models
- Topological properties
  - Manifoldity, Connectedness, Compactness, Self-intersections
  - Physical objects: Classification
  - Physical properties: Surface Area, Volume

PHYSICAL OBJECTS

- Requirements
  Solid models are should to be
  - Rigid
  - Finite
  - Closed under rigid motions and boolean operations
  - Homogeneously 3D
  - Determined by its boundary
  - Have a finite representation

SOLIDS



https://graphics.cg.uni-saarland.de/fileadmin/cguds/courses/ws1112/cg1/rc/Web_Hafner/constructive_solid_geometry.html

- Continuum point set model
  - **R-sets**: bounded, regular, semi-analytic sets

$$S \subset \mathbb{B}^3 := \{x \in \mathbb{R}^3 \mid \|x\| < 1\}$$

$$S = S^* := \overline{\text{int}S}$$

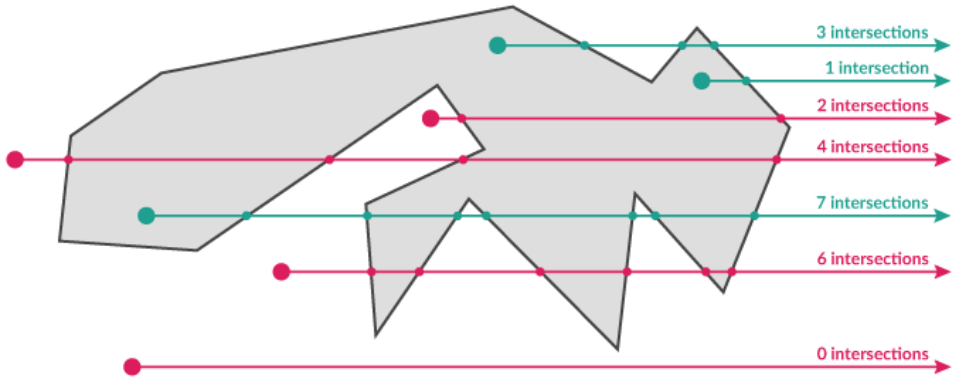
These sets can be defined by boolean operations between following ,,half-spaces''.

$$\{x \in \mathbb{R}^3 \mid f_i(x) \geq 0\}$$

- Combinatorial model
  - **3-manifold** with boundary
  - **R-sets** that are homogeneous n-dimensional topological polyhedra

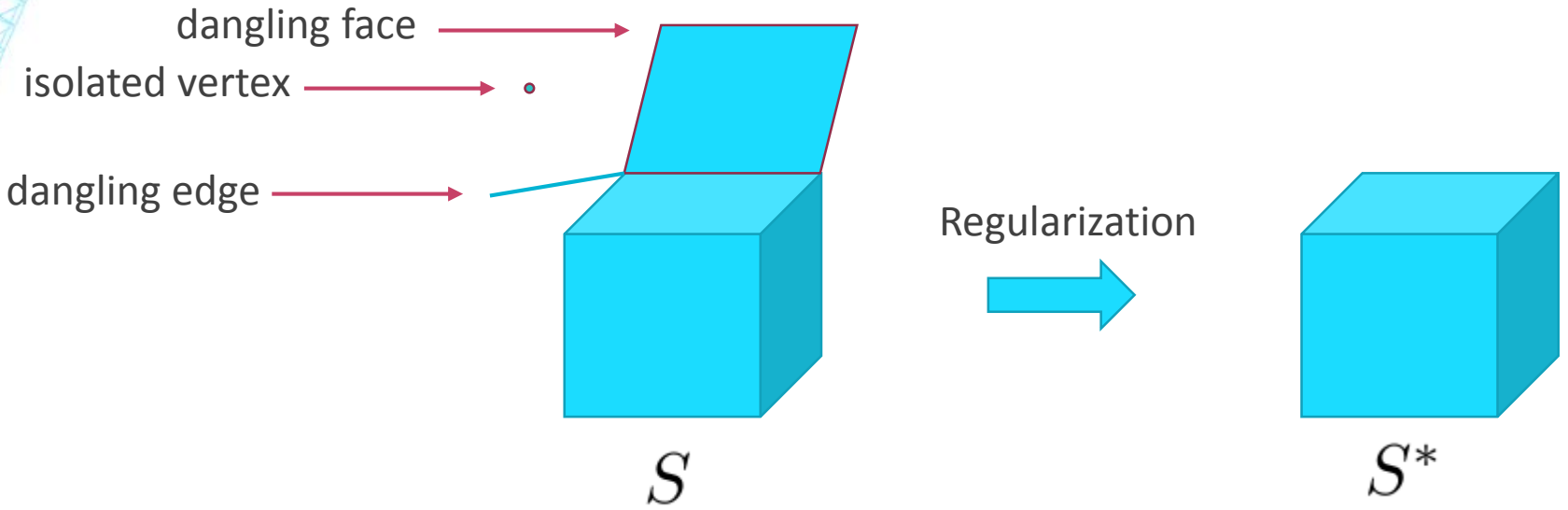
RAY CASTING

- Ray casting works on solids



- Solids the bounded spatial sets where ray casting works?
  - What is the characteristic property?

REGULAR SETS



- Homogeneous 3D sets

- Regularized sets are closed under the regularized boolean operations

$$A \cup^* B := \overline{\text{int}(A \cup B)}$$
$$A \cap^* B := \overline{\text{int}(A \cap B)}$$
$$A \setminus^* B := \overline{\text{int}(A \setminus B)}$$

REGULAR SETS

- **Proposition 1.** Bounded regular sets are compact.
- **Proposition 2.** Non-empty regular set has a positive Lebesgue-measure.
- **Theorem 1**. The boundary of a regular set is well defined.

$$\partial S_1 = \partial S_2 \quad \Rightarrow \quad S_1 = S_2$$

- **Theorem 2.** Ray casting works on regular sets.

$f : [0,1] \to \mathbb{R}^3 \quad x \in \mathrm{int} S$
$f(s) := x + (y - x)s$

$t_1 < t_2 < \cdots < t_n \in (0,1)$
$f \uparrow t_i \downarrow \partial S$

$$\Rightarrow$$

$$\forall j \in [1..n/2] \; \forall s \in (t_{2j-1}, t_{2j}) \; : \; f(s) \in \mathrm{int} S$$

$$\forall j \in [1..n/2] \; \forall s \in (t_{2j}, t_{2j+1}) \; : \; f(s) \in \mathrm{ext} S$$

PHYSICAL OBJECTS I.

- Requirements
  Solid models are should to be
  - Rigid
  - Finite
  - Homogeneously 3D
  - Determined by its boundary
  - Have a finite representation
  - Closed under rigid motions ~~and boolean operations~~
  - Have 2-manifold boundary

PHYSICAL OBJECTS II.

- Requirements
  Solid models are should to be
  - Rigid
  - Finite
  - Closed under rigid motions ~~and boolean operations~~
  - Homogeneously 3D
  - Determined by its boundary
  - Have a finite representation
  - Have a 2-manifold boundary

Rigid, Finite → Bounded subset of 3D space

Homogeneously 3D, Determined by its boundary → Regular

BOUNDARY REPRESENTATION

- Vertexstream & Indexstream

$$V : [1..m] \to \mathbb{R}^3$$
$$F : [1..n] \to [1..m]^3$$

$$[a..b] := [a, b] \cap \mathbb{N}$$
$$m, n \in \mathbb{N}$$

- Surface element

$$\triangle_i := \left\{ x \in \mathbb{R}^3 \mid \quad \alpha^j \in (0,1), \quad \sum_{j=1}^{3} \alpha^j = 1, \quad x = \sum_{j=1}^{3} \alpha^j v_{ij} \right\}$$

$$v_{ij} := V(F(i,j)) = V(f_{ij})$$

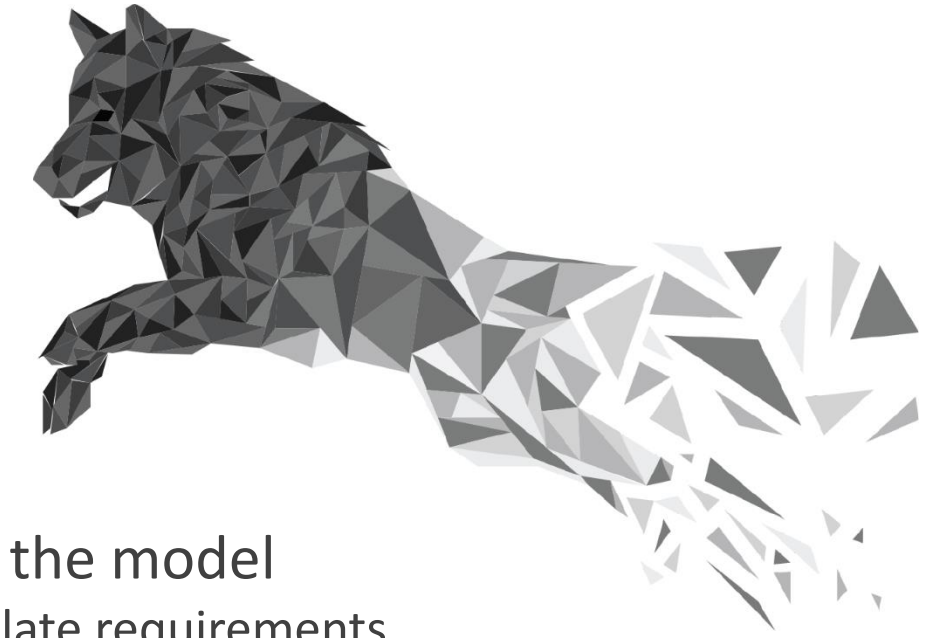
- Model surface = Boundary of a Solid (if exists)

$$\partial S = \overline{\cup_{i=1}^{n} \triangle_i}$$

MESH VS POLYGON SOUP

- (V,F) represents some physical object or that is just an unorganized set of triangles?

http://micromedia.vaniercollege.qc.ca/home/1454243/hypertext/SPA/index.php#wolf



- Analysis of the model
  - Reformulate requirements
  - Mesh repairing or Analysis?

PROBLEM

- Topological information is encoded in streams.

$$V : [1..m] \rightarrow \mathbb{R}^3$$
$$F : [1..n] \rightarrow [1..m]^3$$

- Topological properties
  - Compactness, manifoldity
  - Orientability
  - Connectedness
  - Self-intersection
  - Solidity

- Surface area
- Volume
- Centroid
- …

STREAM ERRORS

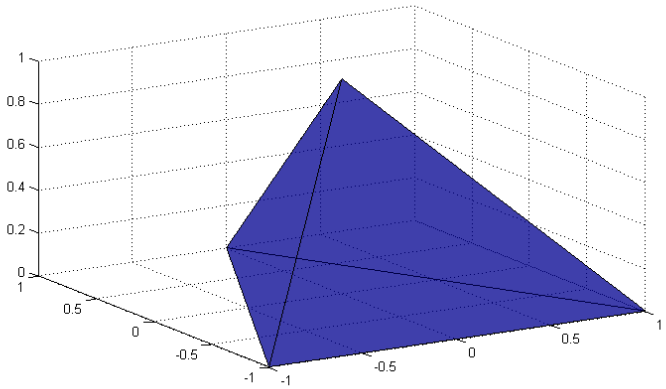
- Vertex stream may contain redundant and deletable elements.

V

1	0	-1
0	0	1
-1	0	-1
0	1	0
0	0	0
0	0	1

F

1	2	3
3	6	4
2	1	4
1	3	4



STREAM ERRORS

- Non-used and redundant vertices have to be removed.

V

1	0	-1
0	0	1
-1	0	-1
0	1	0

F

1	2	3
3	2	4
2	1	4
1	3	4

- The order and index of vertices may be changed.
  - Form equivalence classes
  - Permutation
- Vertex stream without non-used and redundant faces is well defined.

STREAM ERRORS

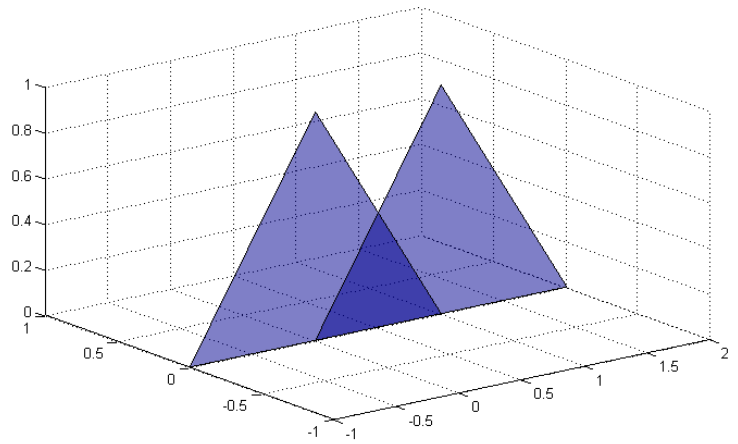
- Index stream may contain empty and overlapping triangles.

V

-1	0	0
1	0	0
0	1	0
0	0	0
2	0	0
1	1	0

F

1	2	3
4	5	6
2	1	1



STREAM ERRORS

- Empty triangles should to be removed, overlapping triangles should to be merged.

V

-1	0	0
1	0	0
0	1	0
0	0	0
2	0	0
1	1	0

F

1	2	3
4	5	6
2	1	1

- [1,2,3] or [4,5,6]?
  - Topological errors can occur. Human supervisor is advised.
  - Mesh repairing is an ill-posed problem.

OPERATIONS

- Matrix-Stream conversion
- Vertex, Face, Edge matrices
- Triadic addition
- defining edges
  calculating face normals
  introducing numerical tolerances

$$\dot{+} : [1..3] \times \mathbb{N} \to [1..3]$$

$$i \dot{+} j := ((i + j - 1) \bmod 3) + 1 \quad (i \in [1..3], j \in \mathbb{N})$$

$$f_{ij} := F(i, j)$$

$$v_{ij} := V(F(i, j)) = V(f_{ij})$$

$$e_{ij} := v_{ij\dot{+}1} - v_{ij}$$

$$n_i := (v_{i2} - v_{i1}) \times (v_{i3} - v_{i1})$$

$$\hat{n}_i := \frac{n_i}{||n_i||}$$

WELL-DEFINED MODEL

- Our model is well-defined, if the following requirements are satisfied.

$$\forall j \in [1..m]\ \exists i \in [1..n]\ \exists l \in [1..3]\ :\ f_{il} = j$$

$$\forall j, j' \in [1..m]\ :\ ||V(j) - V(j')|| \geq \delta_V$$

$$\forall i \in [1..n]\ :\ ||n_i|| \geq \delta_F$$

$$\forall i, i' \in [1..n]\ \ (i \neq i')\ :$$
$$|\langle v_{i'j} - v_{i1}, n_i \rangle| < ||n_i|| \delta_V\ \ (j = 1, 2, 3)\ \Rightarrow\ \triangle_{i'} \cap \triangle_i = \emptyset$$

- The (V,F) pair is unambigously defines ,,something’’.
  - A triangle soup defined by a well-defined model is unique (except permutation of vertex indices).

ADJACENCIES

- We construct an $N \in \mathbb{R}^{m \times m}$ vertex-adjacency matrix. Where N_{ij} is the number of edges in F.
- Construction: Let N be zero matrix. Then for all $i \in [1..n]$ and for all $j \in [1..3]$ let $N_{f_{ij}, f_{ij+1}} := N_{f_{ij}, f_{ij+1}} + 1$
- All topology information needed is encoded in this matrix.
- We can store the adjacencies information more efficiently in an adjacency stream.

MANIFOLDS

- **Theorem 3.** A non-empty well-defined model is a triangulation of some **compact 2-manifold** if and only if

$$\forall i \in [1..m] \; \forall j \in [1..m] \;\; : \;\; N_{ij} + N_{ji} \in \{0, 2\}$$

- **Proposition 3.** Let i and k be indices of two triangles having a common edge defined by f_{ij} and f_{ij+1}. Then the normals of triangles are consistent, if and only if

$$\exists l \in [1..3] \quad f_{kl+1} = f_{ij} \quad f_{kl} = f_{ij+1}$$

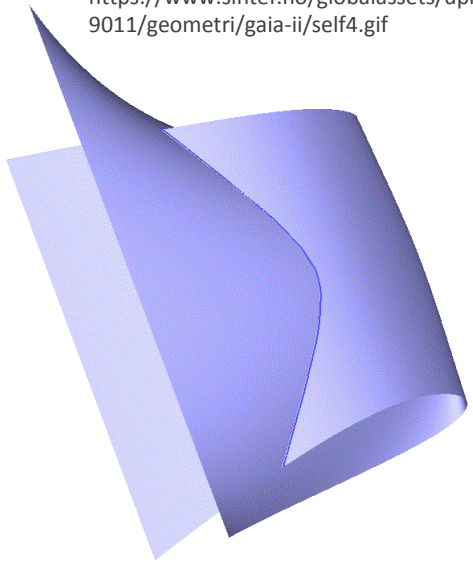
- **Theorem 4.** The compact 2-manifold in **Theorem 2.** is **orientable,** if and only if N is symmetric matrix.

CONNECTEDNESS

- **Proposition 4.** Let N be the adjacency matrix of some directed graph. Then the number of k-length directed walks from node i to node j is $(N^k)_{ij}$.
- **Proposition 5.** In $\mathbb{R}^n$ the connectedness and the path-connectedness are equivalent.
- Therefore, a model is connected if and only if they vertices form a connected graph.
- **Theorem 5.** A well-defined model is connected if and only if

$$\exists k \in \mathbb{N}\ \forall i \in [1..m]\ \forall j \in [1..m]\ :\ (N^k)_{ij} > 0$$

https://www.sintef.no/globalassets/upload/ikt/9011/geometri/gaia-ii/self4.gif



SELF-INTERSECTIONS

- A well defined model is self-intersecting, if

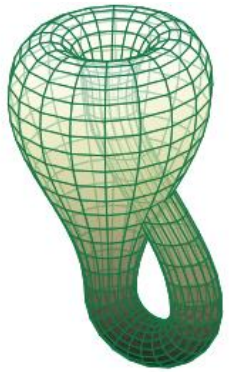
$$\exists i, j \in [1..n]\ (i \neq j)\ :\ \triangle_i \cap \triangle j \neq \emptyset$$

- Self-intersections are not related to adjacencies.
- They can be find by numerous algirithms.
- Simple approach (O(nl)): Search for face-edge intersections.
- The number of edges of a well-defined 2-manifold can be calculated by

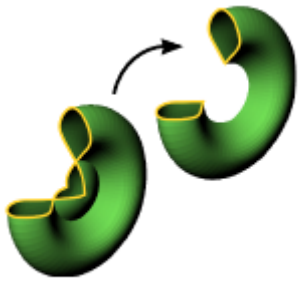
$$l = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} N_{ij} + N_{ji}$$

SOLIDITY

- Does the model represents a physical object?
- The key is the self intersection.
- **Theorem 6.** A well-defined compact 2-manifold model can be embedded into the 3D space, if it is not self-intersecting.
- Opposite direction is not true.



http://www.gnuplotting.org/klein-bottle/

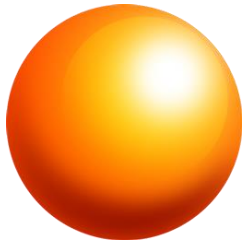


https://www.graphics.rwth-aachen.de/publication/30_35/

CLASSIFICATION

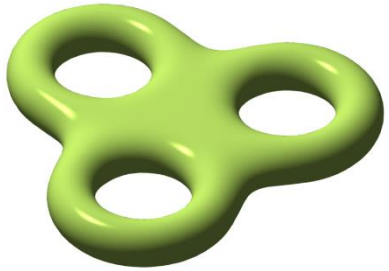
- **Classification theorem of surfaces** states, that all the compact 2-manifolds are topologically equivalent to a sphere or a connected sum of *g* tori.
- **Euler-characteristic** is defined as $\chi = m - l + n$.

$\chi = 2$



https://www.iconfinder.com/icons/62721/ball_orange_sphere_icon

$\chi = 2 - 2g$



https://www.iconfinder.com/icons/62721/ball_orange_sphere_icon

OUR PHYSICAL OBJECTS

- In our approach solids are:
  - well-defined
  - non-self-intersecting
  - orientable
  - compact 2-manifolds
- These definition meets all the requirements of solids.
- Ray casting works on solids, solids have surface area and volume.
- How can we compute?

SURFACE AREA & VOLUME

- Surface area triangle by triangle: $A_i = \frac{1}{2}\|(v_{i2} - v_{i1}) \times (v_{i3} - v_{i1})\|$

$$\sum_{i=1}^{n} A_i$$

- Volume by divergence theorem.

$$\int_S \mathrm{div}\omega \; \mathrm{d}V = \int_{\partial S} \langle \omega, \hat{n} \rangle \; \mathrm{d}A$$

$$\omega(x, y, z) := \tfrac{1}{3}(x, y, z)$$

$$\mathrm{div}\omega = \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = 1$$

$$\int_S dV = \frac{1}{3} \int_{\partial S} (x, y, z)\hat{n} \; \mathrm{d}A =$$

$$\frac{1}{3} \sum_{i=1}^{n} A_i \langle v_{i1}, \hat{n}_i \rangle$$

MATLAB IMPLEMENTATION

- Detecting and removing stream errors
  - Redundant and non-used vertices
  - Empty and overlapping faces
- Looking for self-intersections
- Defining adjacencies
- Checking manifoldity
- Checking orientability
- Determining Euler-characteristic, surface area, volume, centroid, etc.

FUTURE WORKS

- Study some mesh repairing algorithms
- Supervised framework to manipulate meshes safely
- Computing boolean operations on solids properly
- Gluing of manifolds?

THANK YOU FOR YOUR ATTENTION!