Integrals of Scalar Fields over Polygonal Meshes

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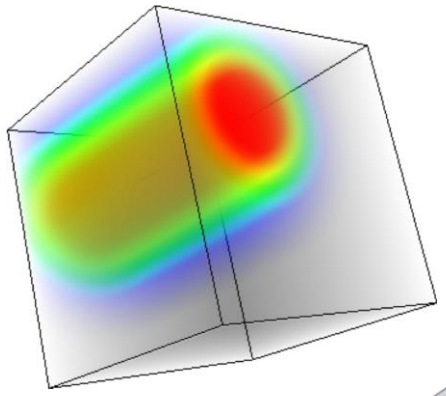
Introduction

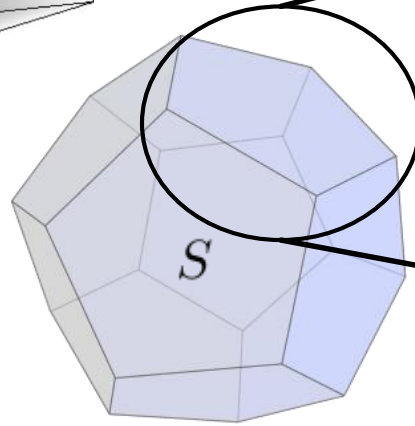
Polygon: Ordered list of 2D or coplanar 3D vertices.

Polygonal Mesh: Unordered list of simple 3D polygons defining a compact 2-manifold.

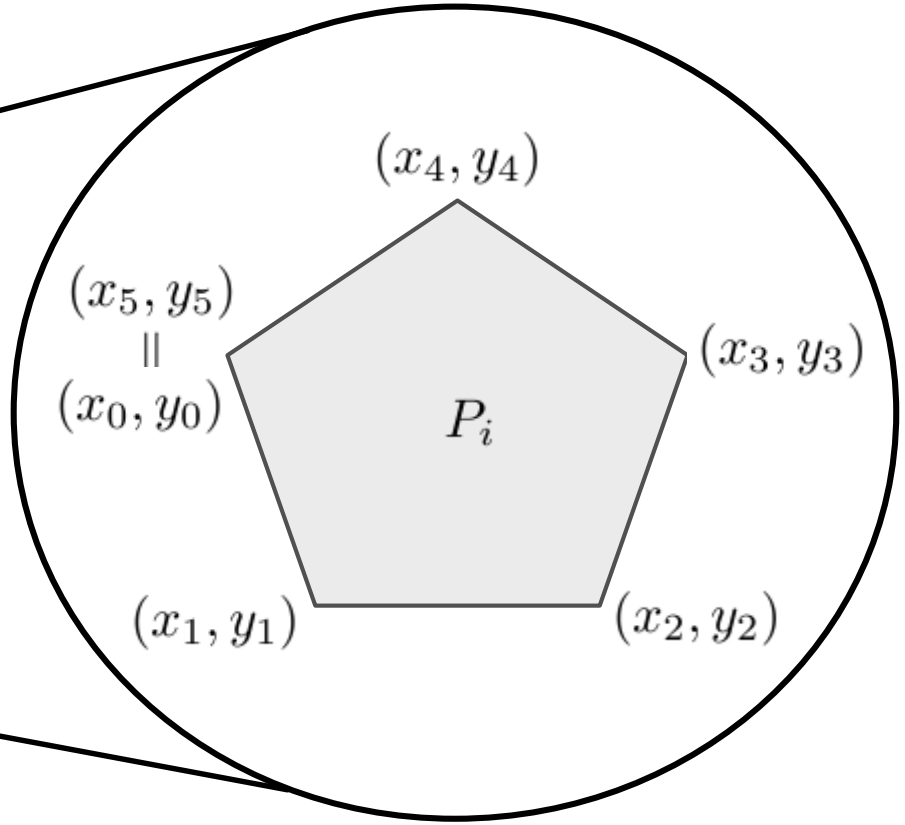
Scalar field: Continuous vector-scalar type function.

Solid: Compact spatial subset with a piecewise linear boundary.





$$\partial S = \bigcup_{i=1}^{N} P_i$$

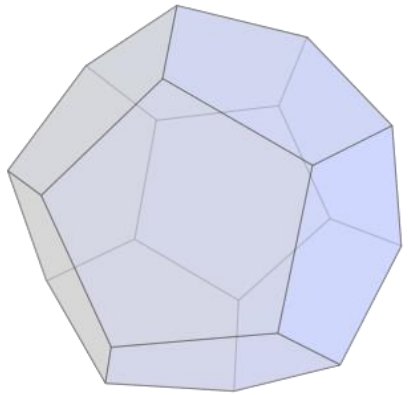


https://simple.wikipedia.org/wiki/Dodecahedron
http://idav.ucdavis.edu/~mhessf/MultiViewTriangulation/multi_view_triangulation.html

Motivation I.

Gauss's divergence theorem

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \iiint_S \operatorname{div}\,\Phi \,\mathrm{d}V = \iint_{\partial S} \langle \Phi, \hat{\mathbf{n}} \rangle \,\mathrm{d}S$$



Divergence theorem on polygonal meshes

$$\phi_i := \langle \Phi, \hat{\mathbf{n}}_i \rangle$$

$$\partial S = \bigcup_{i=1}^{N} P_i \qquad \iiint_S \operatorname{div}\,\Phi \,\mathrm{d}V = \iint_{\partial S} \langle \Phi, \hat{\mathbf{n}} \rangle \,\mathrm{d}S = \sum_{i=1}^{N} \iint_{P_i} \langle \Phi, \hat{\mathbf{n}}_i \rangle \,\mathrm{d}S = \sum_{i=1}^{N} \iint_{P_i} \phi_i \,\mathrm{d}S$$

Physical properties: Volume and Centroid

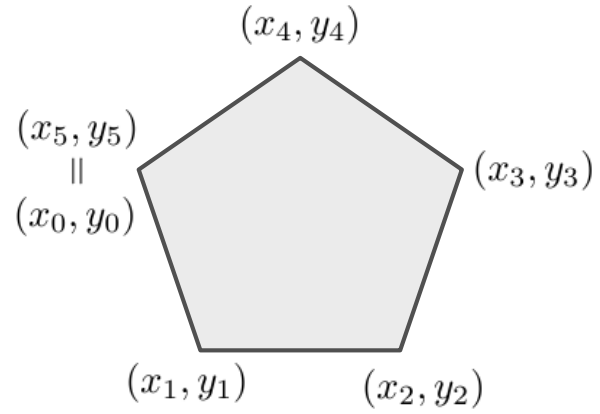
$$\Phi(x,y,z) := \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \operatorname{div}\,\Phi = \frac{\partial}{\partial x} x + 0 + 0 = 1 \qquad \iiint_S \operatorname{div}\,\Phi \,\mathrm{d}V = \iiint_S \mathrm{d}V$$

$$\Phi(x,y,z) := \begin{pmatrix} x^2/2 \\ 0 \\ 0 \end{pmatrix} \qquad \operatorname{div}\,\Phi = \frac{\partial}{\partial x}(x^2/2) + 0 + 0 = x \qquad \iiint_S \operatorname{div}\,\Phi \,\mathrm{d}V = \iiint_S x \,\mathrm{d}V$$

Motivation II.

Stokes' theorem

$$\iint_P \langle \mathbf{curl}\ \mathbf{F}, \hat{\mathbf{n}} \rangle \, \mathrm{d}S = \int_{\partial P} \mathbf{F} \, \mathrm{d}\mathbf{r}$$



Stokes' theorem on simple (planar) polygons

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \ldots, \begin{pmatrix} x_N \\ y_N \end{pmatrix} \qquad l_i = \left\{ \begin{pmatrix} x_i + t(x_{i+1} - x_i) \\ y_i + t(y_{i+1} - y_i) \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t \in [0,1] \right\}$$

$$\partial P = \bigcup_{i=0}^{N-1} l_i \qquad \iint_P \langle \mathbf{curl}\ \mathbf{F}, \hat{\mathbf{n}} \rangle \, \mathrm{d}S = \int_{\partial P} \mathbf{F} \, \mathrm{d}\mathbf{r} = \sum_{i=0}^{N-1} \int_{l_i} \mathbf{F} \, \mathrm{d}\mathbf{r}$$

Problem

$$\langle \mathbf{curl}\ \mathbf{F}, \hat{\mathbf{n}} \rangle = \phi = \langle \Phi, \hat{\mathbf{n}} \rangle \quad \Rightarrow \quad \mathbf{curl}\ \mathbf{F} = \Phi$$

Answer

$$\mathrm{div}\ \Phi = \mathrm{div}\ \mathbf{curl}\ \mathbf{F} \equiv 0$$

Integrals over 2D polygons

Theorem 1.

Let P be a simple planar polygon defined by its vertices:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \ldots, \begin{pmatrix} x_N \\ y_N \end{pmatrix}$$

and let $f : \mathbb{R}^3 \to \mathbb{R}$ contiuous scalar field. Then

$$\iint_{P \subset \mathbb{R}^2} f(x, y, 0)\ \mathrm{d}x\mathrm{d}y = \sum_{i=0}^{N-1} \int_{y_i}^{y_{i+1}} \int_0^{x_i + (\eta - y_i)\frac{x_{i+1} - x_i}{y_{i+1} - y_i}} f(\xi, \eta, 0)\mathrm{d}\xi\mathrm{d}\eta$$

Orthonormal basis on a plane

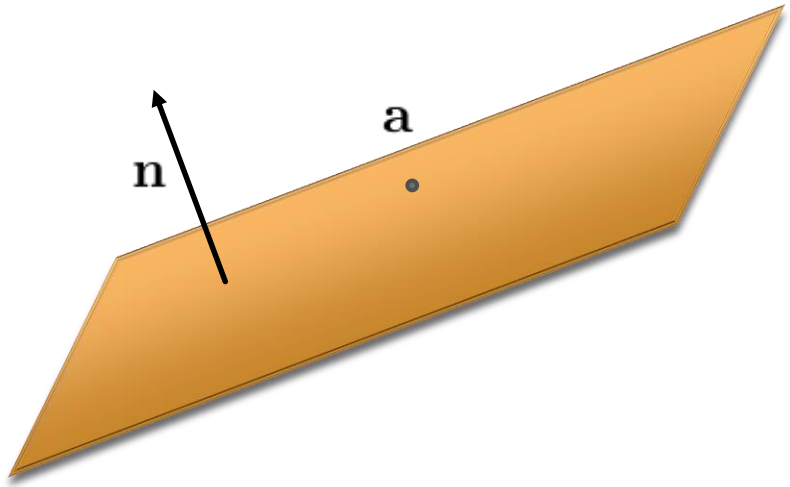
Orthonormal basis

$\mathbf{n}, \mathbf{a} \in \mathbb{R}^3$

$$\omega_1(\mathbf{n}) := \begin{cases} \frac{1}{\sqrt{n_1^2+n_2^2}}(-n_2, n_1, 0) & n_1^2 + n_2^2 \geq n_i^2 + n_j^2 \\ \frac{1}{\sqrt{n_3^2+n_1^2}}(-n_3, 0, n_1) & n_3^2 + n_1^2 \geq n_i^2 + n_j^2 \\ \frac{1}{\sqrt{n_2^2+n_3^2}}(0, -n_3, n_2) & n_2^2 + n_3^2 \geq n_i^2 + n_j^2 \end{cases}$$

$\omega_3(\mathbf{n}) := \hat{\mathbf{n}}$

$\omega_2(\mathbf{n}) := \omega_3(\mathbf{n}) \times \omega_1(\mathbf{n})$

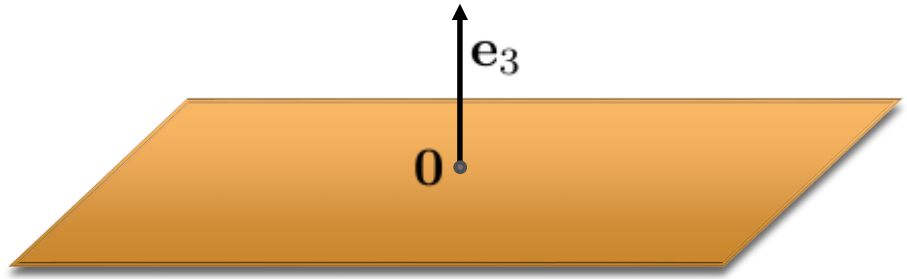


$$P_{\mathbf{n},\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^3 \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{n} \rangle = 0\}$$

Change of basis

$\mathcal{R}_{\mathbf{n},\mathbf{a}} : \mathbb{R}^3 \to \mathbb{R}^3$

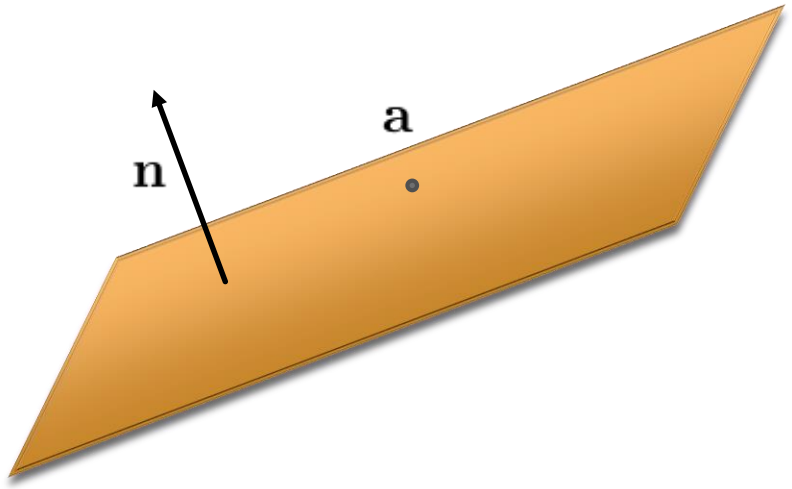
$$\mathcal{R}_{\mathbf{n},\mathbf{a}}(\mathbf{x}) := \sum_{i=1}^{3} \langle \mathbf{x} - \mathbf{a}, \omega_i(\mathbf{n}) \rangle \mathbf{e}_i$$



Rotation between planes

Properties of $\mathcal{R}_{\mathbf{n},\mathbf{a}}$

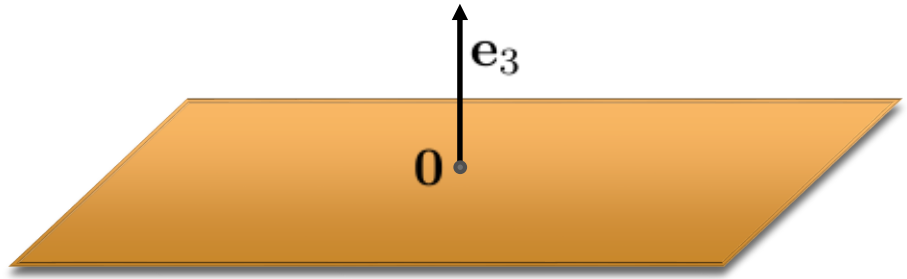
- $\mathcal{R}_{\mathbf{n},\mathbf{a}}[P_{\mathbf{n},\mathbf{a}}] = \mathbb{R}^2 \times \{0\}$
- $\mathcal{R}_{\mathbf{n},\mathbf{a}|P_{\mathbf{n},\mathbf{a}}}$ is bijection between $P_{\mathbf{n},\mathbf{a}}$ and $\mathbb{R}^2 \times \{0\}$
- $\mathcal{R}^{-1}_{\mathbf{n},\mathbf{a}}(\mathbf{x}) = \mathbf{a} + \sum_{i=1}^{3} \langle \mathbf{x}, \mathbf{e}_i \rangle \omega_i(\mathbf{n})$
- $|\det D\mathcal{R}_{\mathbf{n},\mathbf{a}}| = 1$ and $|\det D\mathcal{R}^{-1}_{\mathbf{n},\mathbf{a}}| = 1$



Corollary 1.

$$\iint_{P_{\mathbf{n},\mathbf{a}}} f \, \mathrm{d}S = \iint_{\mathcal{R}_{\mathbf{n},\mathbf{a}}[P_{\mathbf{n},\mathbf{a}}]} (f \circ \mathcal{R}^{-1}_{\mathbf{n},\mathbf{a}}) |\det D\mathcal{R}^{-1}_{\mathbf{n},\mathbf{a}}| \, \mathrm{d}S =$$

$$\iint_{\mathbb{R}^2 \times \{0\}} (f \circ \mathcal{R}^{-1}_{\mathbf{n},\mathbf{a}}) \, \mathrm{d}S = \iint_{\mathbb{R}^2} (f \circ \mathcal{R}^{-1}_{\mathbf{n},\mathbf{a}})(x, y, 0) \, \mathrm{d}x\mathrm{d}y$$



Integrals over 3D polygons

Projection to the plane

$$\mathcal{P}_z : \mathbb{R}^3 \to \mathbb{R}^2 \qquad \mathcal{P}_z(x,y,z) = (x,y)$$

$$\mathcal{P}_z^{-1} : \mathbb{R}^2 \to \mathbb{R}^3 \qquad \mathcal{P}_z^{-1}(x,y) = (x,y,0)$$

$$\overline{\mathbf{x}} = P_z \mathcal{R}_{\mathbf{n},\mathbf{a}}(\mathbf{x}) \in \mathbb{R}^2$$

$$\overline{P} = P_z \mathcal{R}_{\mathbf{n},\mathbf{a}}[P] \subset \mathbb{R}^2$$

Theorem 2.

Let $P \subset P_{\mathbf{n},\mathbf{a}}$ be a simple 3D polygon defined by its vertices:

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \ldots, \begin{pmatrix} x_N \\ y_N \\ z_N \end{pmatrix}$$

and let $f : \mathbb{R}^3 \to \mathbb{R}$ contiuous scalar field. Then

$$\iint_P f(x,y,z)\ \mathrm{d}S = \iint_{\overline{P}} (f \circ \mathcal{R}_{\mathbf{n},\mathbf{a}}^{-1} \circ \mathcal{P}_z^{-1})(\overline{x},\overline{y})\ \mathrm{d}\overline{x}\mathrm{d}\overline{y} =$$

$$\sum_{i=0}^{N-1} \int_{\overline{y}_i}^{\overline{y}_{i+1}} \int_0^{\overline{x}_i + (\eta - \overline{y}_i)\frac{\overline{x}_{i+1}-\overline{x}_i}{\overline{y}_{i+1}-\overline{y}_i}} f(\xi,\eta,0)\mathrm{d}\xi\mathrm{d}\eta$$

Vector fields over solids

$$\iiint_S \operatorname{div}\,\Phi\;\mathrm{d}V = \iint_{\partial S} \langle \Phi, \hat{\mathbf{n}} \rangle\;\mathrm{d}S = \sum_{i=1}^{N} \iint_{P_i} \langle \Phi, \hat{\mathbf{n}}_i \rangle\;\mathrm{d}S = \sum_{i=1}^{N} \iint_{P_i} \phi_i\;\mathrm{d}S$$

scalar fields over **solids**

scalar fields over **polygonal meshes**

Theorem 3.

Let S be a solid defined by a polygonal mesh $\partial S = \bigcup_{i=1}^{N} P_i$. Let Φ be a vector field, and let us suppose, that the partial derivatives of Φ are continuous functions. Then

$$\iiint_S \operatorname{div}\,\Phi\;\mathrm{d}V = \sum_{i=1}^{N} \iint_{P_i} \phi_i\;\mathrm{d}S = \sum_{i=1}^{N} \iint_{\overline{P}_i} (\phi_i \circ \mathcal{R}^{-1}_{\mathbf{n}_i,\mathbf{a}_i} \circ \mathcal{P}_z^{-1})(\overline{x},\overline{y})\;\mathrm{d}\overline{x}\mathrm{d}\overline{y}$$

Integral of polynomials

Why polynomials?

$$\Phi(x,y,z) := \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \operatorname{div}\,\Phi = \frac{\partial}{\partial x}x + 0 + 0 = 1 \qquad \iiint_S \operatorname{div}\,\Phi\,\mathrm{d}V = \iiint_S \mathrm{d}V$$

$$\iiint_S \operatorname{div}\,\Phi\,\mathrm{d}V = \iint_{\partial S} \langle \Phi, \hat{\mathbf{n}} \rangle\,\mathrm{d}S = \sum_{i=1}^{N} \iint_{P_i} \langle \Phi, \hat{\mathbf{n}}_i \rangle\,\mathrm{d}S = \sum_{i=1}^{N} \iint_{P_i} \phi_i\,\mathrm{d}S$$

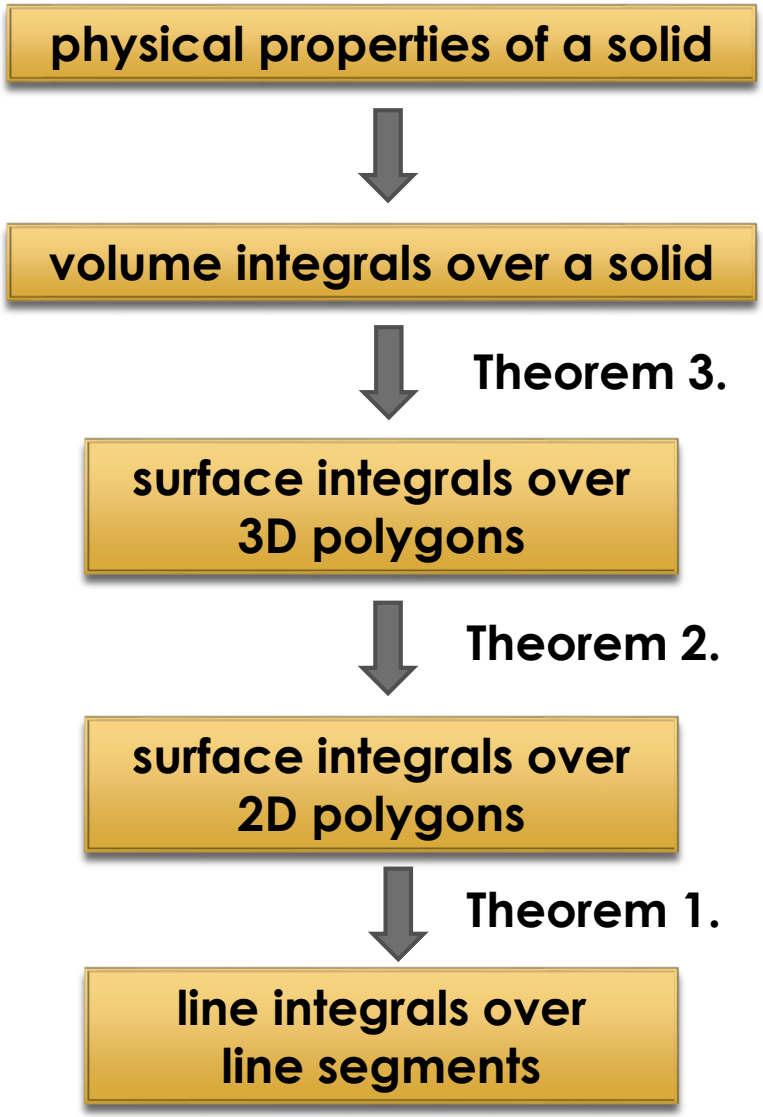
Exact formula

$$f(x,y,0) = \sum_{j=0}^{l}\sum_{k=0}^{l} a_{jk}x^j y^k$$

$$\iint_P f(x,y,z)\,\mathrm{d}S = \sum_{i=0}^{N-1} \int_{\overline{y}_i}^{\overline{y}_{i+1}} \int_0^{\overline{x}_i + (\eta - \overline{y}_i)\frac{\overline{x}_{i+1}-\overline{x}_i}{\overline{y}_{i+1}-\overline{y}_i}} f(\xi,\eta,0)\mathrm{d}\xi\mathrm{d}\eta =$$

$$\sum_{i=0}^{N-1}\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{l=0}^{j+1} \frac{a_j k}{(j+1)(k+l+1)} \binom{j+1}{l} (\overline{x}_i - \alpha_i \overline{y}_i)^{j+1-l} \alpha_i^l (\overline{y}_{i+1}^{k+l+1} - \overline{y}_i^{k+l+1}) \qquad \left(\alpha_i = \frac{x_{i+1}-x_i}{y_{i+1}-y_i}\right)$$

Summary


physical properties of a solid
volume integrals over a solid
Theorem 3.
surface integrals over 3D polygons
Theorem 2.
surface integrals over 2D polygons
Theorem 1.
line integrals over line segments

THANK YOU FOR YOUR ATTENTION!