# OPTIMIZATION IN PERIODIC SPLINE INTERPOLATION

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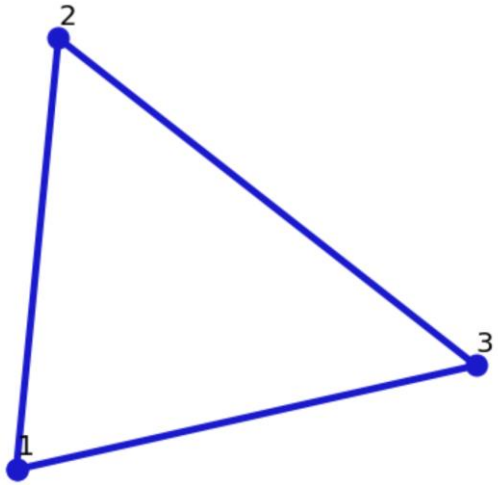
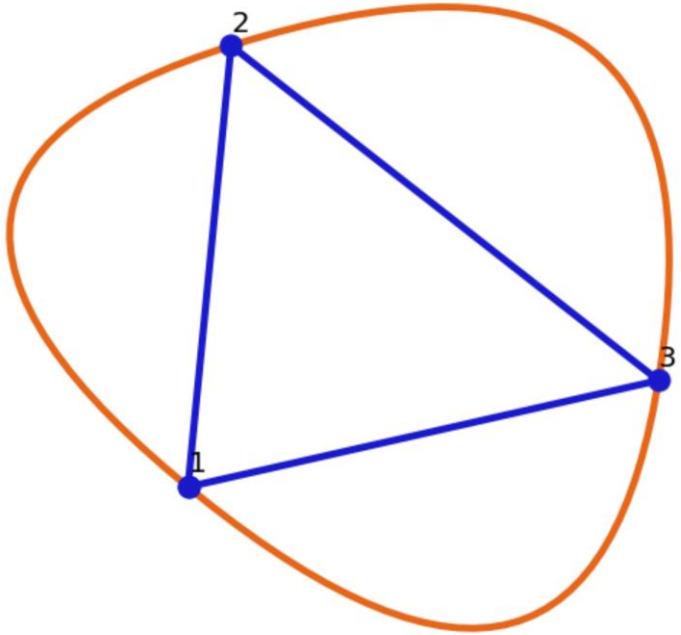
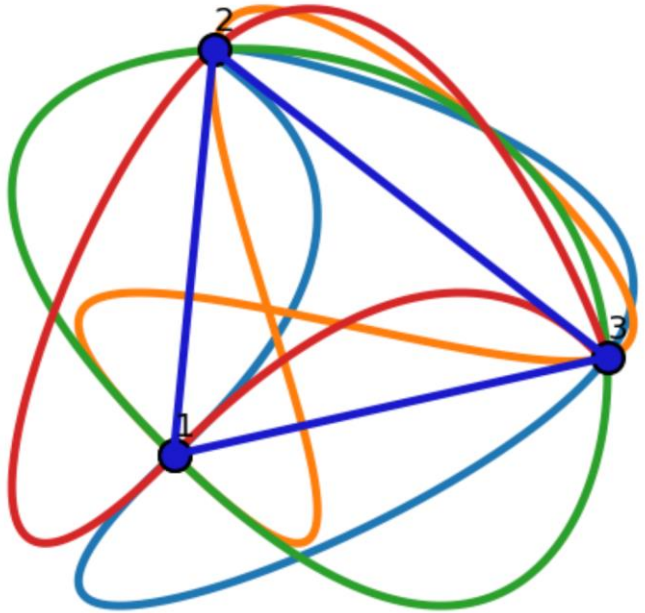
# MOTIVATION



# SPLINE INTERPOLATION

$$f \in C[a,b] \qquad a,b \in \mathbb{R},\ a < b$$

$$a = x_0 < x_1 < \cdots < x_n = b,\ \Omega_n := \{x_0, \ldots, x_n\}$$

$$p_i \in \mathcal{P}_\ell := \left\{ \sum_{k=0}^{\ell} a_k x^k \ \middle|\ a_k \in \mathbb{R} \right\}$$

$$s(x_i) = f(x_i) \qquad (i = 0, \ldots, n)$$

$$s_{|[x_i, x_{i+1}]}(x) := p_i(x) \qquad (i = 0, \ldots, n-1)$$

$$s \in C^{\ell-1}[a,b]$$

$$\Big\} \quad s \in S_\ell(\Omega_n)$$

# VARIABLES VS CONDITIONS

$$s(x) = \begin{cases} p_0(x) = a_{0,0} + a_{0,1}x + \cdots + a_{0,\ell}x^\ell & x \in [x_0, x_1] \\ p_1(x) = a_{1,0} + a_{1,1}x + \cdots + a_{1,\ell}x^\ell & x \in [x_1, x_2] \\ \vdots & \\ p_{n-1}(x) = a_{n-1,0} + a_{n-1,1}x + \cdots + a_{n-1,\ell}x^\ell & x \in [x_{n-1}, x_n] \end{cases}$$

$$\longrightarrow n(\ell + 1)$$

$$\ell - 1$$

BOUNDARY CONDITIONS

$$s_{|[x_i, x_{i+1}]}(x) = p_i(x) \qquad (i = 0, \ldots, n)$$

$$\longrightarrow 2n$$

$$\longrightarrow (\ell - 1)(n - 1)$$

$$s \in C^{\ell-1}[a, b]$$

# CUBIC SPLINE INTERPOLATION

$f \in C[a,b]$
$a,b \in \mathbb{R},\ a < b$

$$a = x_0 < x_1 < \cdots < x_n = b,\ \Omega_n := \{x_0, \ldots, x_n\}$$

$$s_{|[x_i,x_{i+1}]}(x) = p_i(x) \in \mathcal{P}_3 \qquad (i = 0, \ldots, n-1)$$

$$\left.\begin{aligned} s(x_i) &= f(x_i) && (i = 0, \ldots, n) \\ p_i'(x_{i+1}) &= p_{i+1}'(x_{i+1}) && (i = 0, \ldots, n-2) \\ p_i''(x_{i+1}) &= p_{i+1}''(x_{i+1}) && (i = 0, \ldots, n-2) \end{aligned}\right\} s \in C^2(a,b)$$

$$\left.\begin{aligned} & s \in C^2(a,b) \\ & s_{|[x_i,x_{i+1}]}(x) = p_i(x) \in \mathcal{P}_3 \quad (i = 0, \ldots, n-1) \end{aligned}\right\} s \in S_3(\Omega_n)$$

# PERIODIC AND MODIFIED BOUNDARY CONDITIONS

$$a = 0 \quad b = 1 \longrightarrow [a, b) = \mathbb{T}$$

Common periodic boundary conditions

$$s'(x_0) = s'(x_n) \qquad s''(x_0) = s''(x_n) \longrightarrow C^2(\mathbb{T})$$

Modified periodic boundary conditions

$$s'(x_0) = s'(x_n) \qquad s''(x_0) = \beta \longrightarrow C^2(\mathbb{T} \setminus \{0\}) \cap C^1(\mathbb{T})$$

# DETERMINING THE SPLINE (LOCAL PROBLEM)

$$\alpha_0 = p_0'(x_0) \qquad \alpha_n = p_{n-1}'(x_n)$$

$$\alpha_i := p_i'(x_i) = p_{i-1}'(x_i) \qquad (i = 1, \ldots, n-1)$$

$$\beta_0 = p_0''(x_0) \qquad \beta_n = p_{n-1}''(x_n)$$

$$\beta_i := p_i''(x_i) = p_{i-1}''(x_i) \qquad (i = 1, \ldots, n-1)$$

Hermite interpolation problem

$$p_i(x_i) = f(x_i) \quad p_i(x_{i+1}) = f(x_{i+1}) \qquad p_i'(x_i) = \alpha_i \quad p_i''(x_i) = \beta_i$$

# DETERMINING THE SPLINE (LOCAL PROBLEM)

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{\Delta x_i} \qquad \Delta x_i = x_{i+1} - x_i$$

$$p_i(x) = f(x_i) + \alpha_i(x - x_i) + \frac{\beta_i}{2}(x - x_i)^2 + \left(\frac{f[x_i, x_{i+1}] - \alpha_i}{\Delta x_i^2} - \frac{\beta_i}{2\Delta x_i}\right)(x - x_i)^3$$

$$p_i'(x) = \alpha_i + \beta_i(x - x_i) + 3\left(\frac{f[x_i, x_{i+1}] - \alpha_i}{\Delta x_i^2} - \frac{\beta_i}{2\Delta x_i}\right)(x - x_i)^2$$

$$p_i''(x) = \beta_i + 6\left(\frac{f[x_i, x_{i+1}] - \alpha_i}{\Delta x_i^2} - \frac{\beta_i}{2\Delta x_i}\right)(x - x_i)$$

# DETERMINING THE SPLINE (LOCAL PROBLEM)

$$p_i'(x_{i+1}) = \alpha_i + \beta_i \Delta x_i + 3\left(\frac{f[x_i, x_{i+1}] - \alpha_i}{\Delta x_i^2} - \frac{\beta_i}{2\Delta x_i}\right)\Delta x_i^2 =$$

$$3f[x_i x_{i+1}] - 2\alpha_i - \frac{\Delta x_i}{2}\beta_i$$

$$p_i''(x_{i+1}) = \beta_i + 6\left(\frac{f[x_i, x_{i+1}] - \alpha_i}{\Delta x_i^2} - \frac{\beta_i}{2\Delta x_i}\right)\Delta x_i =$$

$$\frac{6}{\Delta x_i} f[x_i x_{i+1}] - 2\beta_i - \frac{6}{\Delta x_i}\alpha_i$$

# DETERMINING THE SPLINE (FORWARD ITERATION)

$$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}) = \alpha_{i+1}$$

$$p_i''(x_{i+1}) = p_{i+1}''(x_{i+1}) = \beta_{i+1}$$

$$\alpha_{i+1} = 3f[x_i x_{i+1}] - 2\alpha_i - \frac{\Delta x_i}{2}\beta_i$$

$$\beta_{i+1} = \frac{6}{\Delta x_i} f[x_i x_{i+1}] - 2\beta_i - \frac{6}{\Delta x_i}\alpha_i$$

# DETERMINING THE SPLINE (RECURSION)

$$A_{i+1}\alpha_{i+1} + B_{i+1}\beta_{i+1} + C_{i+1} = A_i\alpha_i + B_i\beta_i + C_i$$

$$A_{i+1}\alpha_{i+1} + B_{i+1}\beta_{i+1} + C_{i+1} =$$

$$A_{i+1}\left(3f[x_ix_{i+1}] - 2\alpha_i - \frac{\Delta x_i}{2}\beta_i\right) + B_{i+1}\left(\frac{6}{\Delta x_i}f[x_ix_{i+1}] - 2\beta_i - \frac{6}{\Delta x_i}\alpha_i\right) + C_{i+1} =$$

$$(-2A_{i+1} - \frac{6}{\Delta x_i}B_{i+1})\alpha_i + (-\frac{\Delta x_i}{2}A_{i+1} - 2B_{i+1})\beta_i +$$

$$(3A_{i+1}f[x_ix_{i+1}] + \frac{6}{\Delta x_i}B_{i+1}f[x_ix_{i+1}] + C_{i+1})$$

# DETERMINING THE SPLINE (BACKWARD ITERATION)

$$\alpha_n = A_{n-1}\alpha_{n-1} + B_{n-1}\beta_{n-1} + C_{n-1} = \cdots = A_0\alpha_0 + B_0\beta_0 + C_0$$

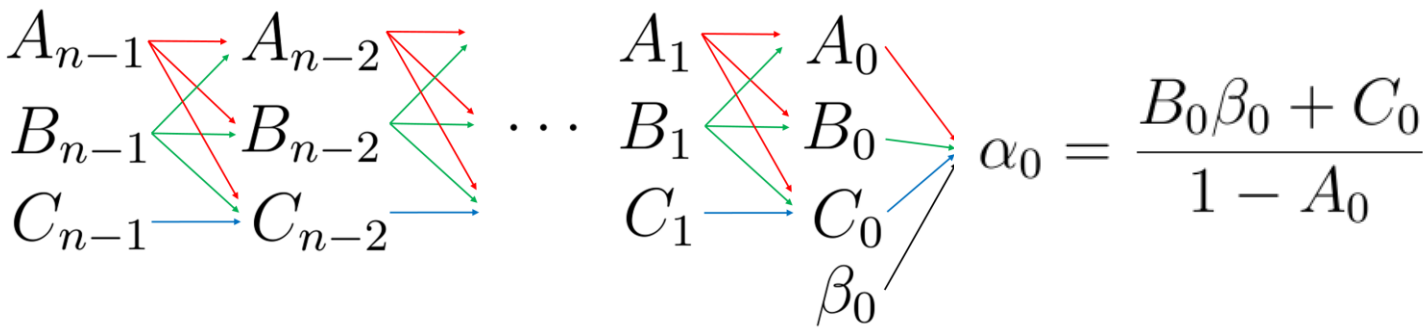
$$\alpha_0 = \frac{B_0\beta_0 + C_0}{1 - A_0}$$

$$\begin{aligned} A_{n-1} &= -2 \\ B_{n-1} &= \frac{\Delta x_{n-1}}{2} \\ C_{n-1} &= 3f[x_{n-1}, x_n] \end{aligned}$$

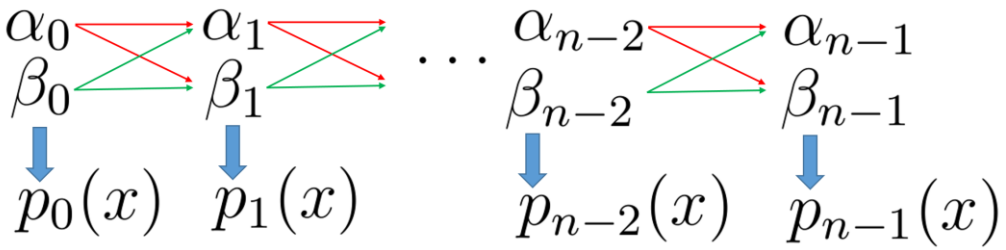
$$\begin{aligned} A_i &= (-2)\left(A_{i+1} + \frac{3}{\Delta x_i}B_{i+1}\right) \\ B_i &= -\frac{\Delta x_i}{2}\left(A_{i+1} + \frac{4}{\Delta x_i}B_{i+1}\right) \\ C_i &= 3(A_{i+1} + \frac{2}{\Delta x_i}B_{i+1})f[x_i x_{i+1}] + C_{i+1} \end{aligned}$$

# DETERMINING THE SPLINE

$$
\begin{aligned}
A_{n-1} &= -2 \\
B_{n-1} &= \frac{\Delta x_{n-1}}{2} \\
C_{n-1} &= 3f[x_{n-1}, x_n]
\end{aligned}
$$




$$\alpha_0 = \frac{B_0\beta_0 + C_0}{1 - A_0}$$



# OPTIMIZATION PROBLEM IN ONE DIMENSION

$$f \in C(\mathbb{T})$$

$$s_\beta \in S_3(\Omega_n) \qquad s'_\beta(x_0) = s'_\beta(x_n) \qquad s''_\beta(x_0) = \beta$$

$$I : S_3(\Omega_n) \to \mathbb{R}$$

$$\beta^* = \operatorname{argmin}_{\beta \in \mathbb{R}} |I(s_\beta)|$$

# OPTIMIZATION PROBLEM IN MULTIPLE DIMENSIONS

$$x_0, x_1, \ldots, x_n \in \mathbb{R}^d \qquad \Gamma : \mathbb{T} \to \mathbb{R}^d \qquad \Gamma(t) = (\Gamma_1(t), \ldots, \Gamma_d(t))$$

$$\Gamma_j(t) := s_{j,\beta_j}(t) \qquad s_{j,\beta_j}(t_i) = (x_i)_j$$

$$I : S_3(\Omega_n) \times \cdots \times S_3(\Omega_n) \to \mathbb{R}$$

$$\mathbb{R}^d \ni \beta^* = \operatorname{argmin}_{(\beta_1,\ldots,\beta_d)\in\mathbb{R}^d} |I(s_{1,\beta_1}, \ldots, s_{d,\beta_d})|$$

# APPLICATION I. MINIMIZING MAXIMUM NORM

$$\|s\|_\infty = \max_{x\in[a,b]} |s(x)| = \max_{i=0}^{n-1} \max_{x\in[x_i,x_{i+1}]} |p_i(x)|$$

$$\gamma_i := \left( \frac{f[x_i, x_{i+1}] - \alpha_i}{\Delta x_i^2} - \frac{\beta_i}{2\Delta x_i} \right)$$

$$\begin{aligned} p_i'(x) &= \alpha_i + \beta_i(x - x_i) + 3\gamma_i(x - x_i)^2 \\ & \alpha_i + \beta_i x - \beta_i x_i + 3\gamma_i x^2 - 6\gamma_i x_i x + 3\gamma_i x_i^2 = \\ & 3\gamma_i x^2 + (\beta_i - 6\gamma_i x_i)x + (\alpha_i + 3\gamma_i x_i^2) \end{aligned}$$

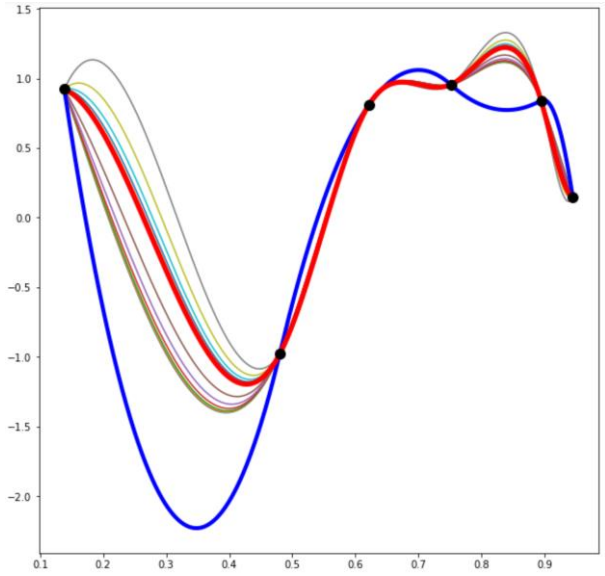
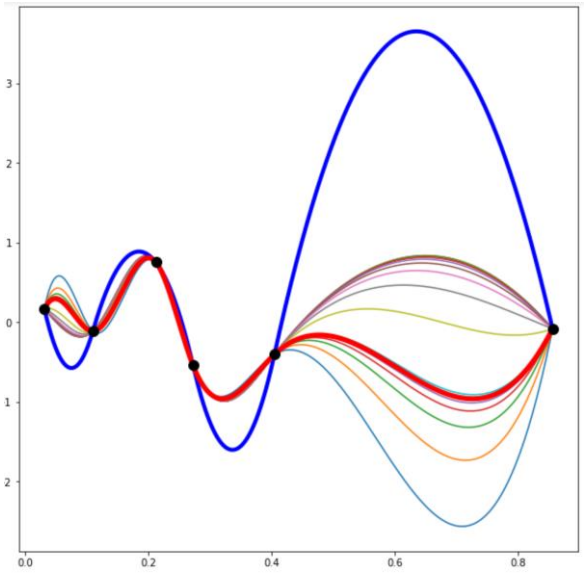
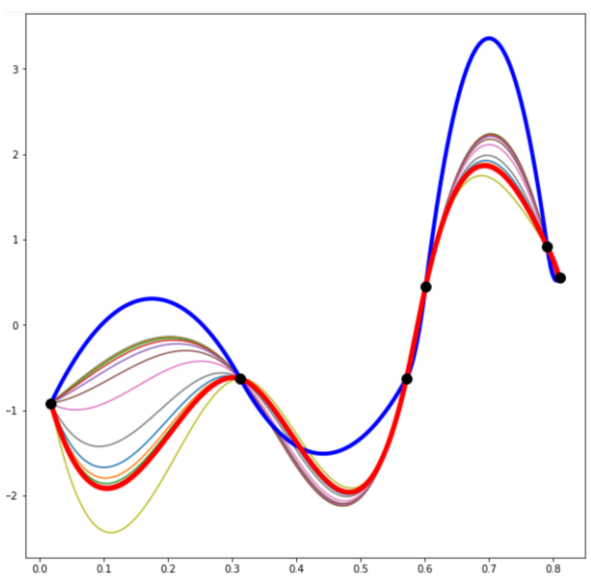
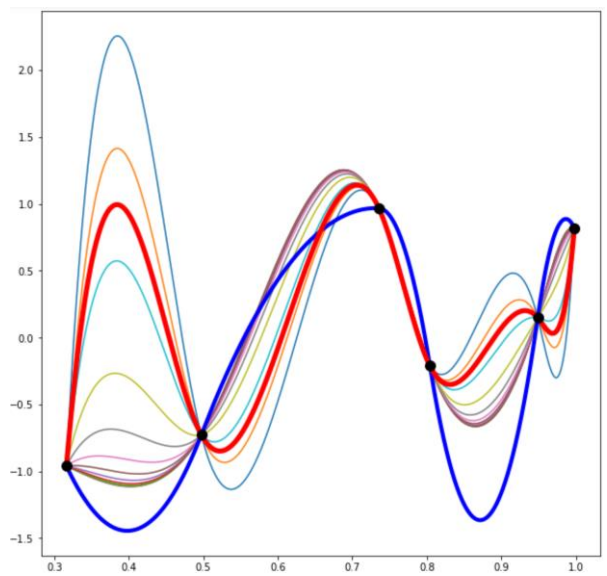
# LOCAL EXTREMA OF THE SPLINE

$$
\begin{aligned}
D_i &= (\beta_i - 6\gamma_i x_i)^2 - 4 \cdot 3\gamma_i \cdot (\alpha_i - \beta_i x_i + 3\gamma_i x_i^2) = \\
&\beta_i^2 - 12\beta_i\gamma_i x_i + 36\gamma_i^2 x_i^2 - 12\alpha_i\gamma_i + 12\beta_i\gamma_i x_i - 36\gamma_i^2 x_i^2 = \\
&\beta_i^2 - 12\alpha_i\gamma_i
\end{aligned}
$$

$$
x_{1,2}^{(i)} = -\frac{\beta_i - 6\gamma_i x_i \pm \sqrt{D_i}}{2 \cdot 3\gamma_i} = x_i - \frac{\beta_i \pm \sqrt{D_i}}{6\gamma_i}
$$

$$
x_{1,2}^{(i)} \in [x_i, x_{i+1}] \quad \Longleftrightarrow \quad 0 \leq -\frac{\beta_i \pm \sqrt{D_i}}{6\gamma_i} \leq \Delta x_i
$$

# APPLICATION I. MINIMIZING MAXIMUM NORM

# APPLICATION II. MINIMIZING ARC LENGTH

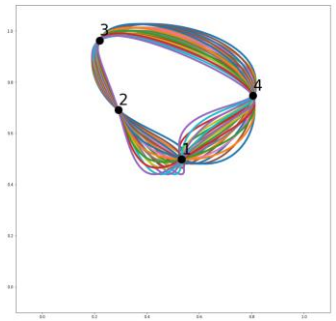
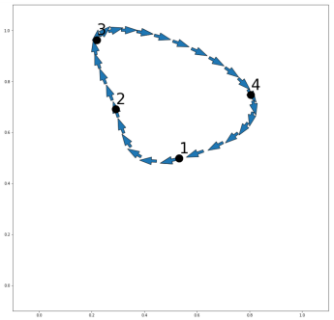
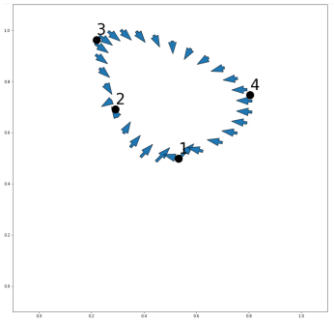
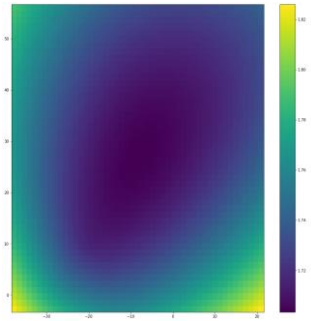
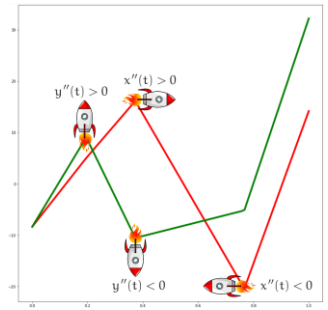
$$(x_0, y_0) = (x_n, y_n) \qquad (x_i, y_i) \to (t_i, x_i), (t_i, y_i) \qquad t_i = \frac{\sum_{j=0}^{i-1} \|x_{i+1} - x_i\|}{\sum_{j=0}^{n-1} \|x_{i+1} - x_i\|}$$

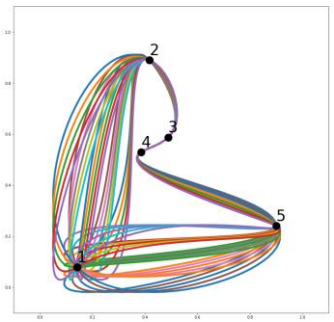
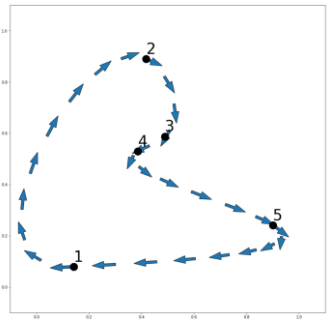
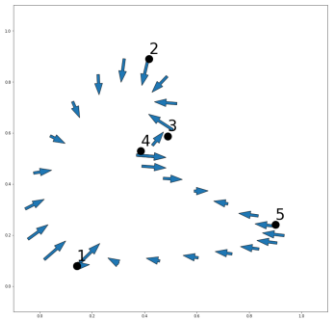
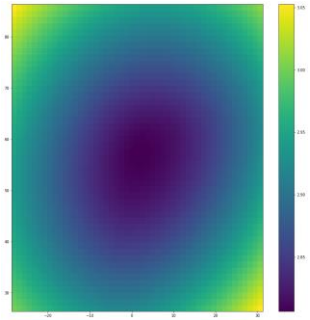
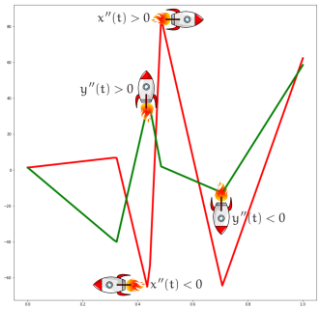
$$\Gamma(t) = (s_{1,\beta_1}(t), s_{2,\beta_2}(t)) \quad s_{j,\beta_j}(t_i) = (x_i)_j \quad L(t) = \sqrt{s_{1,\beta_1}^2(t) + s_{2,\beta_2}^2(t)}$$

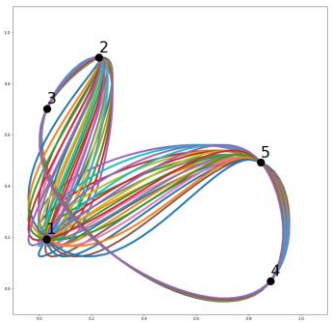
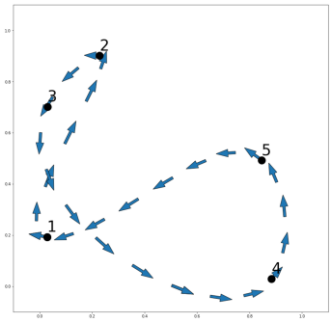
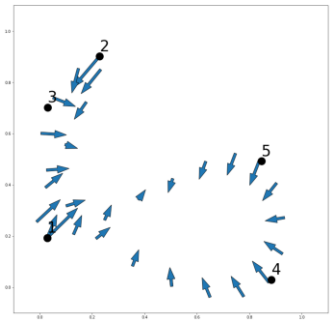
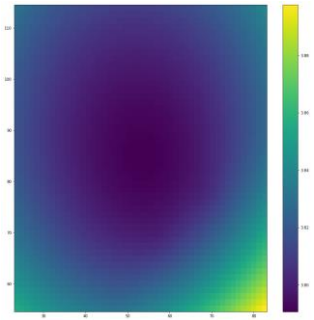
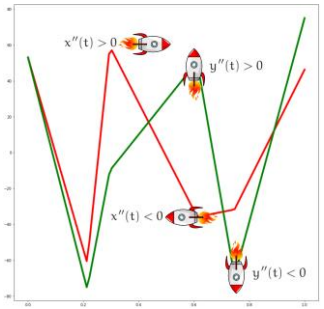
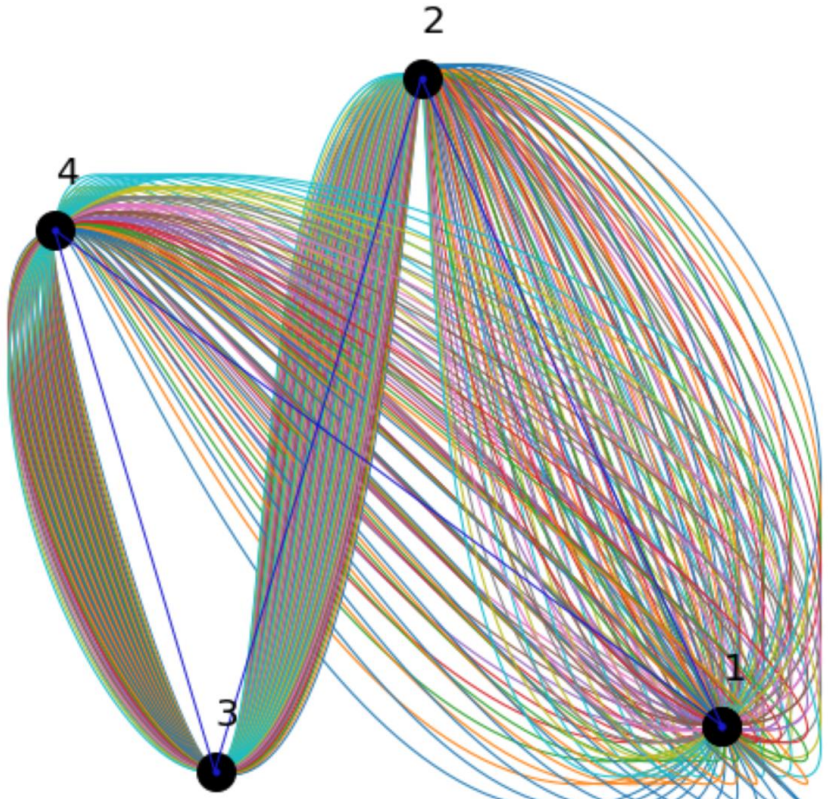
$$\int_0^1 L(t)\ \mathrm{d}t \approx \sum_{i=0}^{M-1} \frac{1}{2M} \left( L(\tfrac{i+1}{M}) + L(\tfrac{i}{M}) \right)$$

$$(\beta_1^*, \beta_2^*) = \operatorname{argmin}_{(\beta_1,\beta_2)\in\mathbb{R}^2} \frac{1}{2M} \sum_{i=0}^{M-1} \left( L(\tfrac{i+1}{M}) + L(\tfrac{i}{M}) \right)$$

# APPLICATION II. MINIMIZING ARC LENGTH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